

We call the frequency of motion of these simple systems the ‘natural’ or ‘resonant’ frequency. This frequency describes the behavior of the system when free from the effects of non-conservative forces. We will now begin to investigate how the system will respond when such forces are introduced.

The first force of concern is friction. We know from our study of periodic physical systems that they tend to lose energy over time and eventually come to rest. The effect of friction is to reduce the velocity of the object by decreasing the acceleration it experiences. This affects both the amplitude of motion and the periodicity.

To study these effects we use a simple harmonic oscillator modeled as a mass-spring system. When there are no external forces acting on the system the equation of motion becomes

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

which has a solution

$$x(t) = x_o \sin(\omega t + \delta)$$

We consider the effects of a velocity dependant frictional or damping term modeled as the product of a damping constant (b) and the instantaneous velocity of the object (dx/dt). Then the equation of motion becomes that of a damped harmonic oscillator

$$\frac{d^2x}{dt^2} - b \frac{dx}{dt} + \frac{k}{m}x = 0$$

and the solution changes accordingly

$$x(t) = x_o e^{\frac{-b}{2m}t} \sin(\omega't + \delta)$$

The damped harmonic oscillator solution produces a system that loses stored energy, with a time dependant term that reduces the amplitude of motion. This reduces the energy

stored within the system and sets a lifetime for the system to oscillate.

Larger damping terms lead to shorter lifetimes. This is accompanied by a change in the angular frequency of the system. The standard expression for mass-spring systems becomes modified by the damping term

$$\omega' = \omega_0 \sqrt{1 - \left(\frac{b}{2m\omega_0}\right)^2}$$

where

$$\omega_0 = \sqrt{\frac{k}{m}}$$

So, in cases where the damping is small, the system behaves almost like an undamped system because the rate of energy loss is not great enough to be noticeable from one oscillation to the next. There exists a damping constant for any system that results in an angular frequency of zero. Such systems are called critically damped and will display one oscillation before returning to rest. Overdamping refers to systems in which the angular frequency becomes imaginary. Such systems do not display any visible oscillatory behavior, completing only a fraction of a cycle before coming to rest.

It is useful at this point to try and establish some parameters with which to compare different systems. Since the only thing that changes the behavior of the system is the ratio of the damping constant and the mass we define the decay time

$$\tau = \frac{m}{b}$$

which is a measure of how rapidly the system loses energy. We define the quality factor

$$Q = \omega_0 \tau$$

and relate it to the fraction of initial system energy lost during each cycle, such that for small values of b

$$Q = \frac{2\pi}{(|\Delta E|/E)_{Cycle}}, (|\Delta E|/E)_{Cycle} \ll 1$$

Thus, a high Q system is one that requires many cycles before exhausting the system energy. The ability of electric circuitry to efficiently deliver power will be rated in a similar fashion in an electromagnetism course.

As power is simply the rate of change of energy in a system, then we are able to examine high Q systems using work and energy considerations. The damping term changes the energy of the system proportional to the speed of the system, so with

$$P = \frac{dE}{dt} = \vec{F}_{Damping} \cdot \vec{v} = -b\vec{v} \cdot \vec{v} = -bv^2$$

and with the idea that the average kinetic energy is half the total system energy over any one cycle we can establish this relationship:

$$\left(\frac{1}{2}mv^2\right)_{Average} = \frac{1}{2}E$$

which leads to the following expression

$$E(t) = E_0 e^{\frac{-b}{m}t} = E_0 e^{\frac{-t}{\tau}}$$

describing the stored system energy as a function of time.

Since the damped oscillator system ‘leaks’ energy, it is necessary to supply energy to the system in order to keep the period and amplitude of motion from changing. When the rate of energy loss is exactly balanced by the rate of energy addition then the system is said to be in a stable state.

If energy is removed from the system faster than it is added, then the system ‘runs down’. On the other hand, if energy is added to the system faster than it is removed, then the amplitude of oscillation will increase.

Any oscillating system that is provided energy by an external source is called a driven harmonic oscillator. As expected, the response of the system depends on the particular nature of the driving force. We will consider a periodic driving force applied to a lightly damped mass-spring system.

If the driving force is of a form

$$F(t) = F_0 \cos(\omega_{Driving}t)$$

then the equation of motion of the system becomes

$$\frac{d^2x}{dt^2} - b\frac{dx}{dt} + \frac{k}{m}x = F_0 \cos(\omega_D t)$$

The general solution of this equation involves a transient solution and a steady state solution. The transient solution reflects the transition of the system from one state to another and has a strong time dependence.

The steady state solution represents the system once the driving force has been applied and the transient effects of the initial motion have diminished. The system can thus be described by the function

$$x(\omega, t) = A_\omega \cos(\omega_D t - \delta)$$

where A is a function of both resonant and driving frequencies:

$$A_\omega = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega_D^2)^2 + b^2\omega_D^2}}$$

The amplitude depends on the difference between driving and system frequencies, reaching a maximum at resonance when the driving and system frequencies are identical.

The appearance of the phase constant δ reflects the synchronicity between the motion of the applied force and the resulting motion of the system. The phase constant is described by the decay time and the difference in the driving and system frequencies.

$$\tan \delta = \frac{b}{m} \frac{\omega_D}{(\omega_0^2 - \omega_D^2)} = \frac{\omega_D}{\tau(\omega_0^2 - \omega_D^2)}$$