Finally, we explore the similarities between uniform circular motion and certain linear systems. We have already seen that motion along a circular path can be written as the sum of displacements in a plane in trigonometry when it was shown that:

$$R^2 = R^2 \sin^2\theta + R^2 \cos^2\theta$$

Thus, the displacement along one axis is given by:

$$R_X = R\cos\theta \text{ or } R_Y = R\sin\theta$$

Using the relationships from rotational motion between the angle displaced, radius and angular speed we are able to rewrite these equations:

$$R_X = R\cos(\omega t) \text{ and } R_Y = R\sin(\omega t)$$

With constant radius, the periodicity of the sin/cos functions shows that the object will return to the same place after a set period of time. It will continue to do so indefinitely, and not only will it return to the same location,

$$x(t+\frac{2\pi}{\omega}) = x_O \cos(\omega(t+\frac{2\pi}{\omega}))$$

$$x(t+\frac{2\pi}{\omega}) = x_O \cos(\omega t + 2\pi)$$

$$x(t+T) = x_O \cos(\omega t)$$

it will do so with the same velocity,

$$v(t) = \frac{dx}{dt} = \frac{d}{dt} x_O \cos(\omega t)$$

$$v(t+T) = -\omega x_O \sin(\omega t)$$

and acceleration

$$a(t) = \frac{dv}{dt} = \frac{d}{dt}(-\omega x_O \sin(\omega t))$$

$$a(t+T) = -\omega^2 x_O \cos(\omega t)$$

The system returns exactly to its initial conditions! This repetitive nature is the definition of a periodic system and is what enables the use of such systems in timekeeping devices.

Of special interest is the relationship between the acceleration and position functions. The acceleration is expressed as a constant multiple of the original displacement function. This will end up as a general behavior of simple periodic systems. For systems to oscillate about a point requires that the system be in a stable equilibrium condition. Such a state is defined by the presence of forces that act on the system in such a way as to oppose changes to the system. Such forces are called restoring forces.

The two most obvious examples of oscillatory behavior are mass-spring systems and the simple pendulum. In each case, the behavior of the system is defined by the potential energy curve that describes the system. Recall the potential energy function of the spring,

$$U(x) = \tfrac{1}{2}kx^2$$

we remember the relationship between force and potential energy,

$$F(x) = \frac{-dU}{dx} = -kx$$

$$ma(x) = -kx$$

$$m\frac{d^2x}{dt^2} = -kx$$

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

This is exactly what was stated as a condition of harmonic motion! The acceleration of the object can be described as a constant multiplying the position function.

Looking at the pendulum, we obtain similar results. Since the potential energy of the pendulum bob can be written as

$$U(\theta) = mgL(1-\cos\theta)$$

we obtain the torques acting on it,

$$\tau(\theta) = \frac{-dU}{d\theta} = -mgL(\sin\theta)$$

$$I\alpha(\theta) = -mgL(\sin\theta)$$

$$\frac{d^2\theta}{dt^2} = \frac{-mgL}{I}\sin\theta \approx \frac{-mgL}{I}\theta$$

$$\frac{d^2\theta}{dt^2} \approx \frac{-g}{L}\theta$$

Again, the acceleration of the object is directly proportional to the position of the object!

When we compare the equations derived from these two systems, we note that each position function is preceded by a term corresponding to the square of the angular frequency ω. From the angular frequency, the period of motion is easily determined. Once again, *the form of the potential energy function determines the response of the system*. This is a point that will be revisited in electricity and magnetism as well as the study of modern physics.