Logarithmic differentiation

Logarithmic differentiation is a technique that is useful for evaluating derivatives of expressions containing combinations of products, quotients, and powers. There are three steps involved. Given $y = f(x)$:

1. Take logs of both sides: $\ln y = \ln f(x)$, and expand the right side using properties of ln.

2. Find $\frac{dy}{dx}$ from this equation by implicit differentiation and solve for $\frac{dy}{dx}$.

3. Substitute $f(x)$ for y, to make $\frac{dy}{dx}$ an expression purely in x.

Examples.

1. Find the derivative of $y = \dfrac{x+1}{x-2}$ by logarithmic differentiation.

Step 1: $$\ln y = \ln\left(\frac{x+1}{x-2}\right) = \ln(x+1) - \ln(x-2)$$

Step 2: $$\frac{d}{dx}\ln y = \frac{d}{dx}[\ln(x+1) - \ln(x-2)]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x+1} - \frac{1}{x-2}$$

$$\frac{dy}{dx} = y\left[\frac{1}{x+1} - \frac{1}{x-2}\right]$$

Step 3: $$\frac{dy}{dx} = \frac{x+1}{x-2}\left[\frac{1}{x+1} - \frac{1}{x-2}\right]$$

A little algebra confirms that this is the same result you would have gotten via the Quotient Rule, or by writing $y = (x+1)(x-2)^{-1}$ and using the Product Rule.

2. Find the derivative of $y = \dfrac{(2x+1)(x+3)^4}{(x+2)^{1/2}}$ by logarithmic differentiation.

Step1: $$\ln y = \ln\left[\frac{(2x+1)(x+3)^4}{(x+2)^{1/2}}\right] = \ln(2x+1) + 4\ln(x+3) - \tfrac{1}{2}\ln(x+2)$$

Step 2: $$\frac{1}{y}\frac{dy}{dx} = \frac{2}{2x+1} + \frac{4}{x+3} - \frac{1}{2}\cdot\frac{1}{(x+2)}$$

Step 3: $$\frac{dy}{dx} = y\left[\frac{2}{2x+1} + \frac{4}{x+3} - \frac{1}{2(x+2)}\right] = \frac{(2x+1)(x+3)^4}{(x+2)^{1/2}}\left[\frac{2}{2x+1} + \frac{4}{x+3} - \frac{1}{2(x+2)}\right]$$