2.2 Definition of derivative

Definition: For a function f, the derivative of f at x is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

ex: Calculate $f'(x)$ for $f(x) = x^2$

Note: This will be like the previous example but with x in place of 1.

Step 0 $\quad f(x) = x^2$

Step 1 $\quad f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$

Step 2 $\quad f(x+h) - f(x) = x^2 + 2xh + h^2 - x^2$
$$= 2xh + h^2$$
$$= h(2x+h)$$

Step 3 $\quad \dfrac{f(x+h) - f(x)}{h} = \dfrac{h(2x+h)}{h} = 2x + h$

↑ average rate of change of f between x and $x+h$ = slope of the secant line

Step 4 $\quad f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

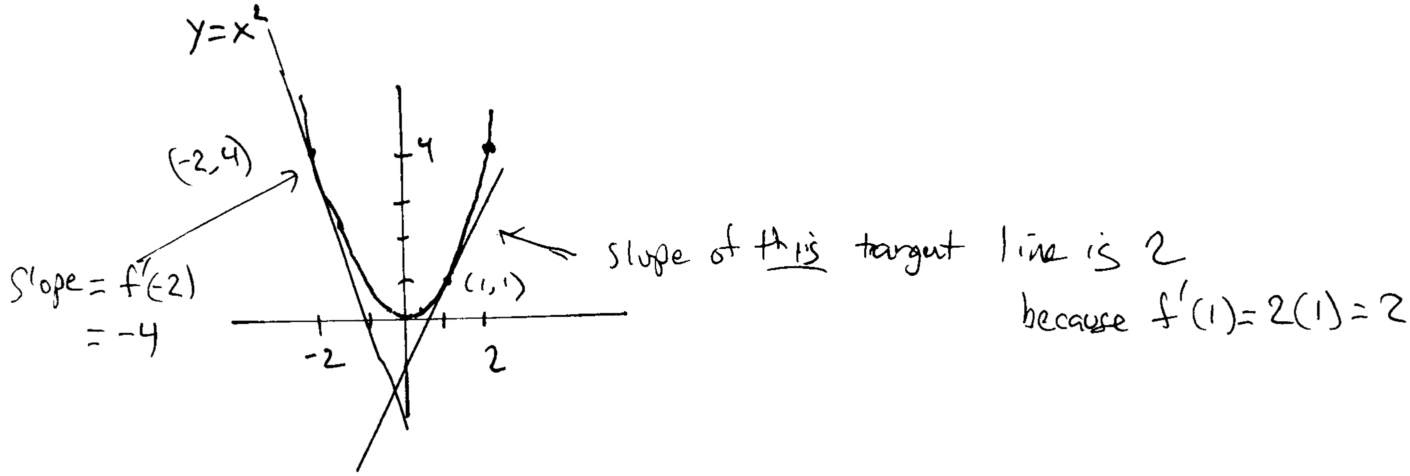
$$= \lim_{h \to 0} 2x + h = \boxed{2x}$$

= instantaneous rate of change

= slope of the tangent line at x

Remark: Note that $f'(x) = 2x$ is *itself* a function.

How is that?



ex: What is the equation of the line tangent to the parabola $y = x^2$ at $(x,y) = (-2,4)$?

Use point-slope form. Point: $(x,y) = (-2,4)$

Slope: $m = f'(-2) = 2(-2) = -4$

$y - y_1 = m(x - x_1)$ becomes

$y - 4 = -4(x - (-2))$

or $y - 4 = -4(x+2)$

or $y - 4 = -4x - 8$

or $y = -4x - 4$

ex: Find $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ for $f(x)=x^4$.

Step 1 $f(x+h) = (x+h)^4$

$= x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$

Numerator of diff. quotient

Step 2 $f(x+h)-f(x) = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4$

$= 4x^3h + 6x^2h^2 + 4xh^3 + h^4$

$= h(4x^3 + 6x^2h + 4xh^2 + h^3)$

Difference quotient

Step 3 $\dfrac{f(x+h)-f(x)}{h} = \dfrac{h(4x^3+6x^2h+4xh^2+h^3)}{h}$

$= 4x^3 + 6x^2h + 4xh^2 + h^3$

Find limit $h \to 0$

Step 4 $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} 4x^3 + 6x^2h + 4xh^2 + h^3$

$= 4x^3 + 0 + 0 + 0$

$= \boxed{4x^3}$

Notation for derivatives If $y = f(x)$

$f'(x) = \dfrac{df}{dx}$

or y' or $\dfrac{dy}{dx}$ ← Reason: $\dfrac{\text{change in } y}{\text{change in } x} = \dfrac{\Delta y}{\Delta x}$

$\lim_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = \dfrac{dy}{dx}$

2.3 Some differentiation formulas

Power Rule $\boxed{\dfrac{d}{dx} x^n = n x^{n-1}}$

That is, if $f(x) = x^n$, then $f'(x) = n x^{n-1}$.

ex: If $f(x) = x^4$ then $f'(x) = 4x^3$.

ex: If $f(x) = x^2$ then $f'(x) = 2x^1 = 2x$.

Q: For what n is this formula valid? Any n.

ex: $y = \sqrt[3]{x}$. Find $\dfrac{dy}{dx}$.

Use exponential notation: $y = x^{1/3}$ (so $n = \frac{1}{3}$ and $n-1 = \frac{1}{3} - 1 = -\frac{2}{3}$)

so $y' = \dfrac{dy}{dx} = \dfrac{1}{3} x^{\frac{1}{3}-1}$

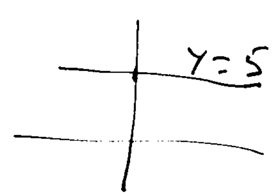
$= \dfrac{1}{3} x^{-\frac{2}{3}} = \dfrac{1}{3\sqrt[3]{x^2}}$

Find $\left.\dfrac{dy}{dx}\right|_{x=8} = f'(8) = \dfrac{1}{3\sqrt[3]{8}^{\,2}} = \dfrac{1}{12}$

Constant Rule $\boxed{\dfrac{d}{dx} c = 0}$

ex: $f(x) = 5$

$f'(x) = 0$



Constant multiple rule $\boxed{\frac{d}{dx}[c \cdot f(x)] = c \cdot f'(x)}$

ex: $\frac{d}{dx}[10x^4] = 10 \cdot \frac{d}{dx}[x^4]$

$= 10 \cdot 4x^3$

$= 40x^3$

Derivative of x: $\boxed{\frac{d}{dx} x = \frac{dx}{dx} = 1}$

Reason: $\frac{d}{dx}[x^1] = 1x^{1-1} = 1x^0 = 1$.

Sum Rule: $\boxed{\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)}$

(there is also a Difference Rule)

ex: $f(x) = 5x^3 + 2x^2 + 7x + 17$ Find $\frac{df}{dx} = f'(x)$.

sum rule $f'(x) = \frac{d}{dx}[5x^3] + \frac{d}{dx}[2x^2] + \frac{d}{dx}[7x] + \frac{d}{dx}[17]$

constant multiple, constant rule $= 5 \cdot \frac{d}{dx}[x^3] + 2\frac{d}{dx}[x^2] + 7\frac{d}{dx}[x] + 0$

power rule $= 5 \cdot 3x^2 + 2 \cdot 2x + 7 \cdot 1$

clean up $= 15x^2 + 4x + 7$