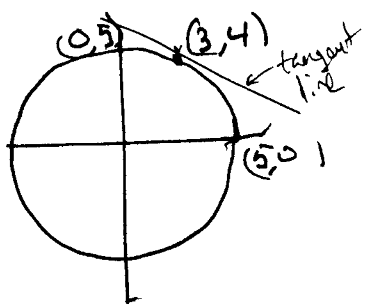
3.6 Implicit Differentiation

ex: Consider $x^2+y^2=25$



Note, the graph fails the vertical line test, so it's not the graph of a function, but pieces of the graph do pass the vertical line test (top semicircle, bottom semicircle).

We say $x^2+y^2=25$ implicitly defines one or more functions of form $y=f(x)$.

[Pretend we don't know that $f(x)=\sqrt{25-x^2}$.]

We still want to find $f'(x)=\frac{dy}{dx}$. Here's how.

$x^2+y^2=25$ and assume x = independent variable

and $y=f(x)$ for some mysterious = dependent variable.

$$\frac{d}{dx}[x^2+y^2]=\frac{d}{dx}[25]$$

$$\frac{d}{dx}[x^2]+\frac{d}{dx}[y^2]=0$$

$$2x+2y\frac{dy}{dx}=0$$

or $2x+2y\cdot y'=0$

By the Chain Rule

Now think of

$$2x + 2yy' = 0$$

as an equation in three variables, Solve for y':

$$2yy' = -2x$$

$$\frac{2y \cdot y'}{2y} = \frac{-2x}{2y}$$

$$\frac{dy}{dx} = \boxed{y' = -\frac{x}{y}}$$

← what is lost by implicit differentiation is: we get an expression in x and y, not just x.

What is the slope of the line tangent to the circle at $(x,y) = (3,4)$?

At $(x,y) = (3,4)$, $\frac{dy}{dx} = \boxed{-\frac{3}{4}}$

ex: Same example, but drop the pretense that we can't solve for y.

$$x^2 + y^2 = 25$$
$$y^2 = 25 - x^2$$
$$y = \pm\sqrt{25-x^2}$$

Top semicircle: $y = \sqrt{25-x^2} = (25-x^2)^{1/2}$

ex (cont'd) Explicit differentiation

If $y = (25 - x^2)^{1/2}$

then $\frac{dy}{dx} = \frac{1}{2}(25 - x^2)^{-1/2} \cdot \frac{d}{dx}[25 - x^2]$

$\frac{dy}{dx} = \frac{1}{2}(25 - x^2)^{-1/2} \cdot (-2x)$

$\frac{dy}{dx} = \frac{-x}{\sqrt{25 - x^2}}$

Is this the same answer?

Previously: $\frac{dy}{dx} = \frac{-x}{y}$

Now: $\frac{dy}{dx} = \frac{-x}{\sqrt{25 - x^2}} = \frac{-x}{y}$

where $y = \sqrt{25 - x^2}$

3.6 (0) $x^2y + xy^2 = 4$ Find $\frac{dy}{dx}$ by implicit differentiation.

$\frac{d}{dx}[x^2y] + \frac{d}{dx}[xy^2] = \frac{d}{dx}[4]$

$2xy + x^2 \cdot \frac{dy}{dx} + \frac{dx}{dx} y^2 + x \cdot \frac{d}{dx}[y^2] = 0$

product Rule

By chain Rule

$$2xy + x^2\frac{dy}{dx} + y^2 + x\cdot 2y\frac{dy}{dx} = 0$$

Now solve for $\frac{dy}{dx}$:

$$x^2\frac{dy}{dx} + 2xy\frac{dy}{dx} = -2xy - y^2$$

$$(x^2+2xy)\frac{dy}{dx} = -2xy-y^2$$

$$\frac{dy}{dx} = \frac{-2xy-y^2}{x^2+2xy}$$

Try this in your notes: Do implicit differentiation to find $\frac{dy}{dx}$

5) $y^4 - x^3 = 2x$

$$\frac{d}{dx}[y^4] - \frac{d}{dx}[x^3] = \frac{d}{dx}[2x]$$

$$4y^3\cdot\frac{dy}{dx} - 3x^2\cancel{\frac{dx}{dx}}^{1} = 2$$

$$4y^3\cdot\frac{dy}{dx} - 3x^2 = 2$$ ← Now done with differentiating

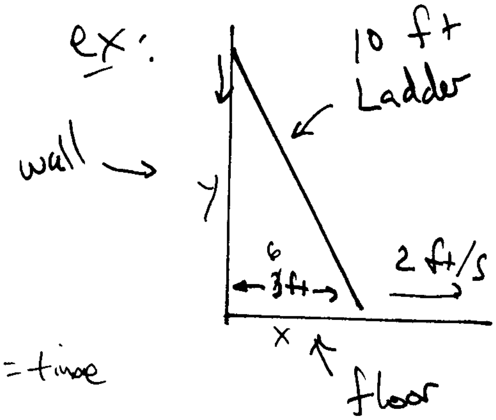
$$4y^3\frac{dy}{dx} = 3x^2+2$$

$$\frac{dy}{dx} = \frac{3x^2+2}{4y^3}$$

Algebra: solve for $\frac{dx}{dx}$.

Related Rates

ex:



t = time

the base of a 10 ft ladder ~~A~~ is being pulled along the floor at 2 ft/s.

At the moment the base is ~~3~~6 feet from the wall, how fast is the top of the ladder moving down the wall.

equ 1 $\quad x^2+y^2=10^2$ ← Pythagorean Thm

But also, x and y are each functions of t.

we want info about $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{d}{dt}\left[x^2+y^2\right]=\frac{d}{dt}[100]$$

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$$

equ 2 $\quad x\frac{dx}{dt}+y\frac{dy}{dt}=0$

what we know: $\frac{dx}{dt}=2$ ft/s when $x=6$ ft

Also when $x=\cancel{3}6$ ft, $\quad 6^2+y^2=10^2$

$$36+y^2=100$$
$$y^2=64$$
$$y=8 \text{ ft}$$

$$\underbrace{(6\,ft)(2\,ft/s)}_{12} + (8\,ft)\frac{dy}{dt} = 0$$

$$8\frac{dy}{dt} = -12$$

$$\frac{dy}{dt} = -\frac{12}{8} = -\frac{3}{2} = -1.5\ ft/s$$

The top of the ladder is <u>dropping</u> at 1.5 ft/s at the moment in question.