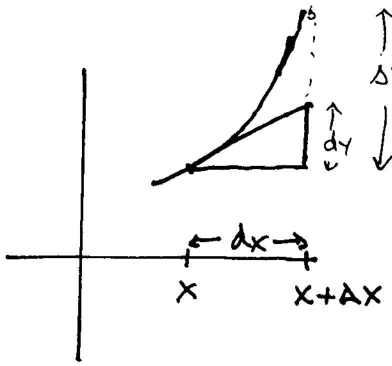
3.7 Differentials and approximations



Remark: We've seen

$$\frac{dy}{dx} = f'(x)$$

but we've never seen just dy.

Define: $dx = \Delta x$

Then $dy = f'(x)\,dx$

That is, rise $= \left(\frac{\text{rise}}{\text{run}}\right) \cdot$ run

slope of tangent line at x.

Remark: Why this is useful: when $\Delta x = dx$ is small

Δy can be approximated by dy ← easy to calculate

usually needs a calculator

ex [Practice problem #1 p231] For $y = x^2 + 3$,

find dy and evaluate it at $x = 3$ and $dx = -0.05$.

$$dy = f'(x)\,dx = \left(\frac{dy}{dx}\right) dx = 2x\,dx$$

That is

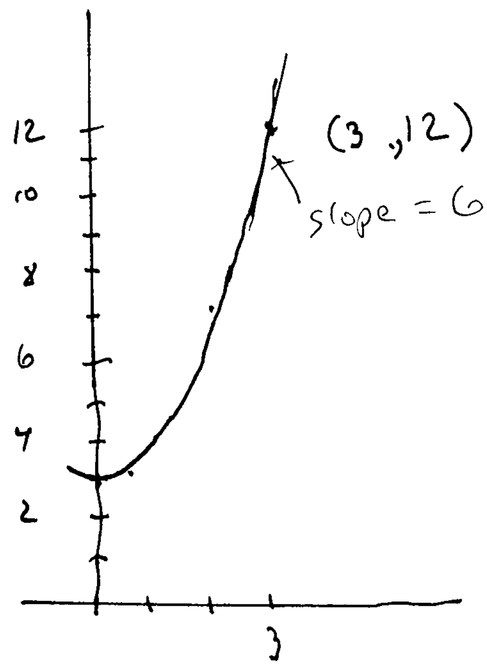
$$dy = 2x\,dx \quad \text{when } y = x^2 + 3$$

Now let $x = 3$ and $dx = -0.05$. Then what is dy?

$$dy = 2(3) \cdot (-0.05) = 6 \cdot (-0.05) = -0.3$$

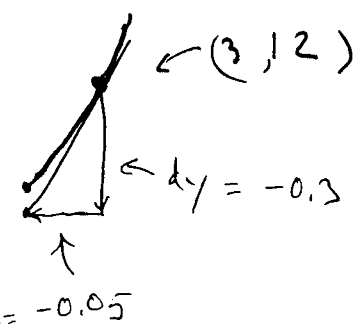
Interpretation: Graph $y = x^2 + 3$

when $x = 3$, $y = 3^2 + 3 = 12$



because $f'(3) = 2(3) = 6$

zoom in



Using the tangent line to approximate the graph.

we would say $\Delta y = f(x + \Delta x) - f(x)$

$$\Delta y = f(3 + -.05) - f(3)$$
$$= f(2.95) - f(3)$$

Idea: $\Delta y \approx dy$ so

$$-0.3 \approx f(2.95) - f(3)$$
$$f(2.95) \approx f(3) + dy$$
$$= f(3) + (-0.3)$$
$$= 12 + (-0.3)$$
$$= 11.7$$

That is we're estimating $f(2.95)$ to approx. equal 11.7.

The actual answer: $f(2.95) = 2.95^2 + 3 = 11.7025$

Linear approximation

idea: $\Delta y \approx dy$

That is, $\Delta y = f(x+\Delta x) - f(x)$

is approximated by $dy = f'(x)\,dx$

so $f(x+\Delta x) - f(x) \approx f'(x)\,dx$

so $\boxed{f(x+\Delta x) \approx f(x) + f'(x)\,dx}$

ex: 18) Approximate $\sqrt{48}$ by a good choice of x and Δx.

Take x close to 48 but easy to calculate

so take $x = 49$, and let $f(x) = \sqrt{x}$

Then choose Δx so that $x + \Delta x = 48$

$$49 + \Delta x = 48$$
$$\Delta x = 48 - 49 = -1$$

$$f(48) = f(x+\Delta x) \approx f(x) + f'(x)\,dx$$
$$= f(49) + \tfrac{1}{2}(49^{-1/2})\cdot(-1)$$
$$= \sqrt{49} + \tfrac{1}{2}\cdot\frac{1}{\sqrt{49}}\cdot(-1)$$
$$= 7 + \tfrac{1}{14}\cdot(-1)$$
$$= 7 - \tfrac{1}{14} = 6\tfrac{13}{14} = 6.92857$$

Note: If $f(x) = x^{1/2}$
$f'(x) = \frac{1}{2}x^{-1/2}$

Actual value: $\sqrt{48} = 6.92820$

25) $R(x) = 5x^2 - 35x + 104$

Find dR and ΔR for $x = 2$, $dx = \Delta x = 0.4$

$$dR = R'(x)\,dx = (10x - 35)\,dx$$

when $x = 2$ and $dx = 0.4$

$$dR = (10 \cdot 2 - 35)(0.4) = (-15)(0.4) = -6$$

$$\Delta R = R(x + \Delta x) - R(x)$$
$$= R(2 + 0.4) - R(2)$$
$$= R(2.4) - R(2)$$
$$= 48.8 - 54 = -5.2$$