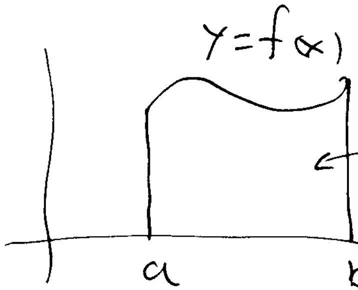
5.3 Fundamental Theorem of Calculus (cont'd)

Last Time: $\int_a^b f(x)\,dx = \left\{ \begin{array}{l} \text{area under the} \\ \text{curve } y=f(x) \end{array} \right\}$



area $= \int_a^b f(x)\,dx$

Fundamental Theorem of Calculus

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F(x)$ is any antiderivative of f.

ex: $\int_1^3 3x^2\,dx = ?$

Step 1 Find an antiderivative of $3x^2$

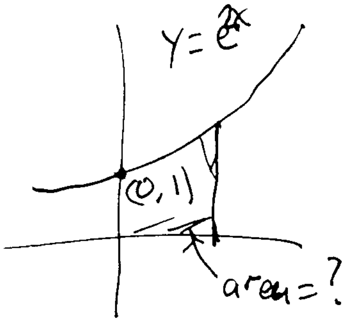
$$\int 3x^2\,dx = x^3 + C. \text{ Take } F(x) = x^3.$$

Step 2 $\int_1^3 3x^2\,dx = F(3) - F(1) = F(x)\Big|_1^3$

$$= 3^3 - 1^3 = 27 - 1$$

$$= \boxed{26}$$

ex: Find the area under the curve $y = e^{2x}$, between $x=0$ and $x=1$.



$$\text{area} = \int_0^1 e^{2x}\,dx$$

$$= \frac{1}{2} e^{2x} \Big|_0^1$$

$$= \frac{1}{2} e^{2} - \frac{1}{2} e^{0} = \frac{1}{2} e^{2} - \frac{1}{2}$$

47)

$$\int_0^1 \left(x^{99} + x^{9} + 1 \right) dx$$

$$= \left(\frac{1}{100} x^{100} + \frac{1}{10} x^{10} + x \right) \Big|_0^1$$

$$= \left(\frac{1}{100} \cdot 1^{100} + \frac{1}{10} \cdot 1^{10} + 1 \right) - (0 + 0 + 0)$$

$$= \frac{1}{100} + \frac{1}{10} + 1 = \frac{1 + 10 + 100}{100} = \frac{111}{100}$$

5.6 Integration by Substitution

= antiderivatives + chain rule

Recall <u>Differentials</u>: (Section 3.7)

$$\frac{df}{dx} = f'(x) \quad \text{so} \quad df = f'(x)\,dx$$

ex: If $f(x) = x^3$, $df = 3x^2\,dx$

If $y = \ln x$, $dy = \frac{1}{x}\,dx$

If $y = e^{3x}$, $dy = 3e^{3x}\,dx$

If $u = 5x^2+1$, $du = 10x\,dx$.

ex: [Integration by substitution]

$$\int \overbrace{(x^5+5)}^{u}{}^{10} \cdot \underbrace{5x^4\,dx}_{du}$$

Let $u = x^5+5$

$du = 5x^4\,dx$

$$= \int u^{10}\,du$$

$$= \frac{1}{11}u^{11} + C$$

$$= \boxed{\frac{1}{11}(x^5+5)^{11} + C}$$

<u>Remark</u>: One check that $\frac{d}{dx}\frac{1}{11}(x^5+5)^{11} = (x^5+5)^{10}\cdot 5x^4$ by the Chain Rule.

ex [Substitution, adjust by a constant multiple]

$$\int x^3\sqrt{x^4+1}\,dx = \int (x^4+1)^{1/2}\cdot\frac{1}{4}\cdot 4x^3\,dx$$

Try $u = x^4+1$

Then $du = 4x^3\,dx$

$$= \frac{1}{4}\int (x^4+1)^{1/2}\cdot 4x^3\,dx$$

$$= \frac{1}{4}\int u^{1/2}\,du$$

$$= \frac{1}{4}\cdot\frac{2}{3}\,u^{3/2} + C$$

$$= \frac{1}{6}\,u^{3/2} + C$$

$$= \frac{1}{6}\,(x^4+1)^{3/2} + C$$

Rules of thumb for choosing u:

1. Let u = (stuff in parenthesis)
2. Let u = denominator in $\frac{\text{top}}{(\text{denom})}$
3. Let u = exponent in e^{exponent}