5.6 Integration by Substitution (a bit more)

How to choose u in substitution: Rules of Thumb

① Let $u =$ (stuff in parentheses)

example 30) $\int (3y^2-6y)^3 (y-1)\, dy = (*)$

Try letting $u = 3y^2-6y$

Then (we have no choice) $du = (6y-6)\,dy$
$= 6(y-1)\,dy$

$$(*) = \frac{1}{6} \int \underbrace{(3y^2-6y)^3}_{u^3} \cdot \underbrace{6(y-1)\,dy}_{du}$$

$$= \frac{1}{6} \int u^3\, du = \frac{1}{6} \cdot \frac{1}{4} u^4 + C$$

$$= \frac{1}{24} u^4 + C = \frac{1}{24} (3y^2-6y)^4 + C$$

(2) Let $u =$ exponential expression in $e^{\text{exp. expression}}$.

ex 16) $\int e^{-x^4} x^3 \, dx = -\frac{1}{4} \int e^{-x^4} \cdot (-4x^3) \, dx$

Try $u = -x^4$
then $du = -4x^3 dx$

$= -\frac{1}{4} \int e^u \, du$

$= -\frac{1}{4} e^u + C$

$= -\frac{1}{4} e^{-x^4} + C$

(3) Let $u =$ denominator in $\frac{\text{numerator}}{\text{denominator}}$

ex 42) $\int \frac{e^{3x}}{e^{3x} - 1} \, dx = \frac{1}{3} \int \frac{3e^{3x} dx}{e^{3x} - 1}$

Try $u = e^{3x} - 1$
$du = 3e^{3x} \, dx$

$= \frac{1}{3} \int \frac{du}{u} = \frac{1}{3} \int \frac{1}{u} du$

$= \frac{1}{3} \ln|u| + C$

$= \frac{1}{3} \ln|e^{3x} - 1| + C$

Remark: These three rules of thumb correspond to:

(1) $\int u^n \, du = \frac{1}{n+1} u^{n+1} + C$

(2) $\int e^u \, du = e^u + C$

(3) $\int \frac{du}{u} = \ln|u| + C$

36) $\int \frac{x^2 - x}{(2x^3 - 3x^2)^3}\, dx = \frac{1}{6} \int \frac{6(x^2-x)}{(2x^3-3x^2)^3}\, dx$

$= \frac{1}{6} \int \frac{du}{u^3}$

$= \frac{1}{6} \int u^{-3}\, du$

$= \left(\frac{1}{6}\right)\left(\frac{-1}{2}\right) u^{-2} + C$

$= \frac{-1}{12} u^{-2} + C$

$= -\frac{1}{12} (2x^3-3x^2)^{-2} + C$

$= \frac{-1}{12(2x^3-3x^2)^2} + C$

<u>Try</u> (this will fail)

$u = (2x^3-3x^2)^3$

$du = 3(2x^3-3x^2)^2 \cdot (6x^2-6x)$

No way to match up expression for u and du

Now Try

$u = 2x^3 - 3x^2$

$du = (6x^2 - 6x)\, dx$

$= 6(x^2-x)\, dx$

Substitution in definite integrals

ex: 54) $x=3 \rightarrow$ (implied) $x=2 \rightarrow$

$$\int_2^3 \frac{x^2}{x^3-7}\,dx = \int_{x=2}^{x=3} \frac{x^2}{x^3-7}\,dx$$

Try $u = x^3-7$

$du = 3x^2dx$

$x=2 \Leftrightarrow u = 2^3-7 = 1$

$x=3 \Leftrightarrow u = 3^3-7 = 20$

$$= \frac{1}{3}\int_{x=2}^{x=3} \frac{3x^2dx}{x^3-7}$$

$$= \frac{1}{3}\int_{u=1}^{u=20} \frac{du}{u}$$

$$= \frac{1}{3}\ln|u|\Big|_{u=1}^{u=20}$$

$$= \frac{1}{3}\ln 20 - \cancel{\frac{1}{3}\ln 1}^{\,0}$$

$$= \frac{1}{3}\ln 20$$

Remark: when you change variables in the integrand and in the limits, in definite integrals, you need not change back to x. The answer will be a number (like all definite integrals).

Review of Chapters 2-4

Know Differentiation Rules: (1) Product Rule
(2) Quotient "
(3) Chain "

Basic (1) $\frac{d}{dx} x^n = n x^{n-1}$

(2) $\frac{d}{dx} e^{ax} = a e^{ax}$

(3) $\frac{d}{dx} \ln x = \frac{1}{x}$

Definition: of derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

ex: Show, using the definition only, of derivative that if $f(x) = x^2 + x$, then $f'(x) = 2x+1$.

Step 1: $f(x+h) = (x+h)^2 + (x+h)$
$= x^2 + 2xh + h^2 + x + h$

Step 2: $f(x+h) - f(x) = x^2 + 2xh + h^2 + x + h$
$-x^2 \qquad -x$
$= 2xh + h^2 + h$
$= h(2x + h + 1)$

Step 3: $\frac{f(x+h) - f(x)}{h} = 2x + h + 1$

Step 4: $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} 2x + h + 1 = 2x + 0 + 1$
$= 2x + 1$

ex. $\lim_{x \to 2} \frac{x+2}{x^2-4} = \lim_{x \to 2} \frac{x-2}{(x-2)(x+2)}$

$= \lim_{x \to 2} \frac{1}{x+2} = \frac{1}{4}$

ex implicit differentiation

3.5 26) Find $\frac{dy}{dx}$ at $x=1, y=-1$ if

$$y^2+y+1 = x$$

$$\frac{d}{dx}[y^2+y+1] = \frac{d}{dx}[x]$$

$$2y\frac{dy}{dx} + \frac{dy}{dx} + 0 = 1$$

$$(2y+1)\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{2y+1}$$

$$\left.\frac{dy}{dx}\right|_{(x,y)=(1,-1)} = \frac{1}{2(-1)+1} = \frac{1}{-1} = -1$$

ex 19) p303 Find $f'(x)$ if $f(x) = \ln \sqrt[3]{x}$.

$$f(x) = \ln x^{1/3} = \frac{1}{3}\ln x$$

so $f'(x) = \frac{1}{3} \cdot \frac{d}{dx}\ln x = \frac{1}{3} \cdot \frac{1}{x} = \frac{1}{3x}$

Moral: Use properties of logs