5.6 Integration by Substitution (cont'd)

ex [Substitution in definite integrals]

54) $$\int_{x=2}^{x=3} \frac{x^2}{x^3-7}\,dx = \frac{1}{3}\int_{x=2}^{x=3} \frac{3x^2\,dx}{x^3-7}$$

$u = x^3 - 7$
$du = 3x^2\,dx$
$x=2 \Leftrightarrow u = 2^3 - 7 = 1$
$x=3 \Leftrightarrow u = 3^3 - 7 = 20$

$$= \frac{1}{3}\int_{u=1}^{u=20} \frac{du}{u}$$

$$= \frac{1}{3} \ln|u| \Big|_{u=1}^{u=20}$$

$$= \frac{1}{3}\ln 20 - \frac{1}{3}\ln 1 = \boxed{\frac{1}{3}\ln 20}$$

Remark: By changing the variable in the limits as well as the integrand in a definite integral, we don't need to change back to the original variables.

56) $$\int_0^3 \sqrt{x^2+16}\; x\,dx = \frac{1}{2}\int_{x=0}^{x=3} (x^2+16)^{1/2}\cdot 2x\,dx$$

$u = x^2 + 16$
$du = 2x\,dx$
$x=0 \Leftrightarrow u = 16$
$x=3 \Leftrightarrow u = 25$

$$= \frac{1}{2}\int_{u=16}^{u=25} u^{1/2}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_{u=16}^{u=25}$$

$$= \frac{1}{3}\cdot 25^{3/2} - \frac{1}{3}\cdot 16^{3/2} = \frac{1}{3}\cdot 125 - \frac{1}{3}\cdot 64$$

$$= \frac{1}{3}(125-64) = \frac{1}{3}\cdot 61 = \boxed{\frac{61}{3}}$$

"Basic" integral formula versus "Modified Basic" integral formula

exemples

$$\int e^x\,dx = e^x + C \qquad\qquad \int e^{3x+5}\,dx = \frac{1}{3}e^{3x+5} + C$$

$$\int \frac{1}{x}\,dx = \ln|x| + C \qquad\qquad \int \frac{1}{3x+5}\,dx = \frac{1}{3}\ln|3x+5| + C$$

$$\int x^{1/2}\,dx = \frac{2}{3}x^{3/2} + C \qquad\qquad \int (3x+5)^{1/2}\,dx$$

$$= \frac{1}{3}\cdot\frac{2}{3}(3x+5)^{3/2} + C$$

How you could prove the last "modified basic" example:

ex: $$\int (3x+5)^{1/2}\,dx = \frac{1}{3}\int (3x+5)^{1/2}\cdot 3\,dx$$

$u = 3x+5$

$du = 3\,dx$

$$= \frac{1}{3}\int u^{1/2}\,du$$

$$= \frac{1}{3}\cdot\frac{2}{3}u^{3/2} + C$$

$$= \frac{1}{3}\cdot\frac{2}{3}(3x+5)^{3/2} + C$$

$$= \frac{2}{9}(3x+5)^{3/2} + C$$

Basic	Modified Basic
1) $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C$	(1') $\int (ax+b)^n\,dx = \dfrac{1}{a}\cdot\dfrac{(ax+b)^{n+1}}{n+1} + C$
(2) $\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$	(2') $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln\lvert ax+b\rvert + C$
(3) $\int e^x\,dx = e^x + C$	(3') $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C$

ex: $\displaystyle\int\left(\frac{1}{\sqrt{2x-1}} + e^{5x} + \frac{1}{7x+4}\right)dx$

$$= \int\left[(2x-1)^{-1/2} + e^{5x} + (7x+4)^{-1}\right]dx$$

$$= \frac{1}{2}\cdot 2(2x-1)^{1/2} + \frac{1}{5}e^{5x} + \frac{1}{7}\ln\lvert 7x+4\rvert + C$$

$$= \sqrt{2x-1} + \frac{1}{5}e^{5x} + \frac{1}{7}\ln\lvert 7x+4\rvert + C$$

6.1 Integration by Parts

= antiderivatives + product rule

Main idea

product Rule: $\frac{d}{dx}[uv] = u' \cdot v + u \cdot v'$

Antiderivative form:

$$uv = \int (u'v + uv')\,dx$$
$$= \int u'v\,dx + \int uv'\,dx$$
$$= \int v\,du + \int u\,dv$$

Solve for the last term:

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Ugly integral   Nice integral

u = part to be differentiated
dv = part to be integrated

ex: $\int x e^{3x}\,dx$

$$= (x)\left(\tfrac{1}{3}e^{3x}\right) - \int \tfrac{1}{3}e^{3x}\,dx$$

Scratch work

$u = x$ $\quad dv = e^{3x}\,dx$

$du = dx$ $\quad v = \frac{1}{3}e^{3x}$

no "+C" needed; take C=0.

$$= (x)\left(\tfrac{1}{3}e^{3x}\right) - \tfrac{1}{3}\cdot\tfrac{1}{3}e^{3x} + C$$

$$= \tfrac{1}{3}xe^{3x} - \tfrac{1}{9}e^{3x} + C$$

Remark: To check this you will use the product rule.

ex: $\int 2x \ln x \, dx = (\ln x)(x^2) - \int (x^2)\left(\frac{1}{x}\right) dx$

$u = \ln x \qquad dv = 2x\,dx$

$du = \frac{1}{x}\,dx \qquad v = x^2$

$= x^2 \ln x - \int x\,dx$

$= x^2 \ln x - \frac{x^2}{2} + C$