6.3 Improper Integrals

Limits as x approaches $\pm\infty$

ex: $\lim\limits_{x \to \infty} \dfrac{1}{x^2} = 0$

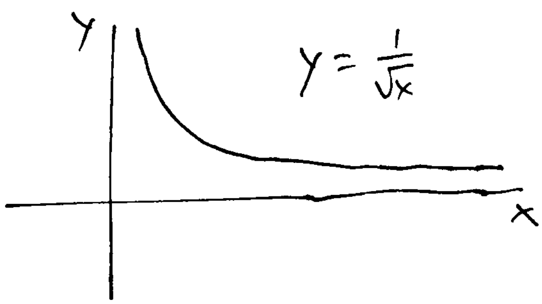
idea: $\dfrac{1}{(\text{HUGE})^2} \approx 0$

$\dfrac{1}{(10^6)^2} = 10^{-12} = \dfrac{1}{1\,000\,000\,000}$

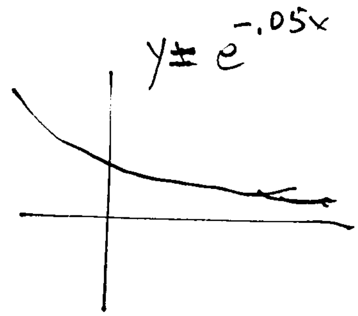
= almost zero.

$\lim\limits_{x \to \infty} \dfrac{1}{x^n} = 0$ provided $n > 0$

ex: $\lim\limits_{x \to \infty} \dfrac{1}{\sqrt{x}} = \lim\limits_{x \to \infty} \dfrac{1}{x^{1/2}} = 0$



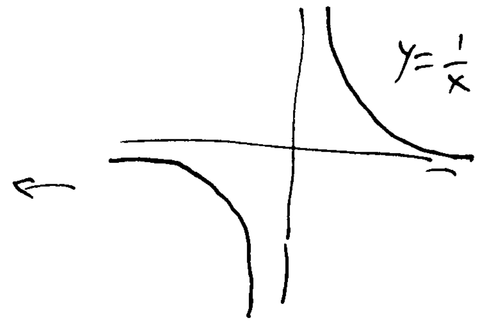
ex: $\lim\limits_{x \to \infty} e^{-.05x} = 0$



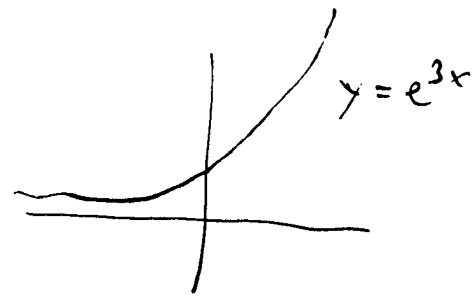
e.g. $e^{-.05(1000)} \approx 2 \cdot 10^{-22}$

e.g. $e^{-.05(20)} \approx 0.368$

ex: $\lim_{x \to -\infty} \frac{1}{x} = 0$



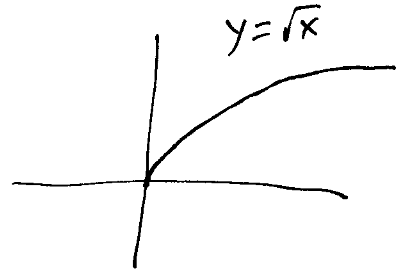
ex: $\lim_{x \to -\infty} e^{3x} = 0$



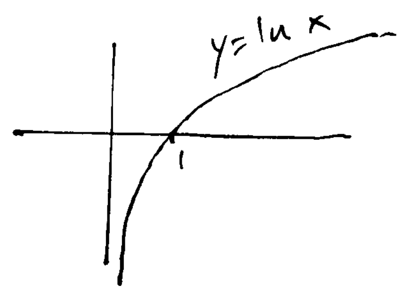
ex: $\lim_{x \to \infty} e^{3x}$ does not exist

ex: $\lim_{x \to \infty} \sqrt{x}$ does not exist

$= \lim_{x \to \infty} x^{1/2}$



ex: $\lim_{x \to \infty} \ln x$ does not exist

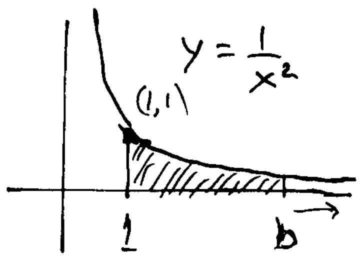


ex: $\lim_{b \to \infty} \left(1 - \frac{1}{b^2} + e^{-b}\right)$

$= 1 - \lim_{b \to \infty} \frac{1}{b^2} + \lim_{b \to \infty} e^{-b} = 1 - 0 + 0$

$= 1$

ex: $$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx$$



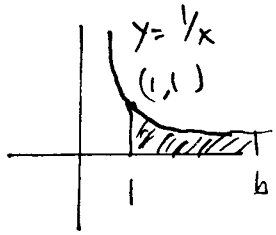
$$\int_1^b x^{-2}\,dx$$

$$= -x^{-1}\Big|_1^b$$

$$= -b^{-1} + 1^{-1} = \frac{-1}{b} + 1$$

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \left[\frac{-1}{b} + 1\right] = -0 + 1 = \boxed{1}$$

ex: $$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx$$



$$\int_1^b \frac{1}{x}\,dx = \ln|x|\Big|_1^b = \ln|b| - \ln 1$$

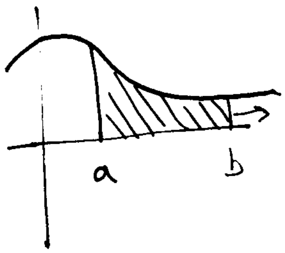
$$= \ln b - 0$$

$$= \ln b$$

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \ln b$$ This limit does not exist.

The improper integral $\boxed{\text{DIVERGES}}$.

Defn: If f is continuous and nonnegative for $x \geq a$ we define



$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

provided this limit exists.

If the limit exists, the integral is convergent

If the limit does not exist, the integral is divergent.

ex: [Exple 5] The present value of an income stream of \$2000 per year (at 5% interest compounded continuously) is

$$\int_{t=0}^{t=\infty} 2000\, e^{-0.05t}\,dt = \lim_{b\to\infty} \int_0^b 2000\, e^{-0.05t}\,dt$$

Definite integral step:

$$\int_0^b 2000\, e^{-0.05t}\,dt = \frac{2000}{-0.05}\, e^{-0.05t}\Big|_0^b$$

$$= -40{,}000\, e^{-0.05t}\Big|_0^b$$

$$= -40{,}000\, e^{-0.05b} + 40{,}000\, \underbrace{e^{0}}_{\text{equals } 1}$$

Limit step:

$$\lim_{b\to\infty} -40{,}000\, e^{-0.05b} + 40{,}000 = 0 + 40{,}000$$

$$= 40{,}000 \text{ dollars}$$

30) [Substitution plus improper integral]

$$\int_0^\infty \frac{x^2}{x^3+1}\,dx = \lim_{b\to\infty} \int_0^b \frac{x^2}{x^3+1}\,dx$$

$$\frac{1}{3}\int_0^b \frac{3x^2}{x^3+1}\,dx = \frac{1}{3}\int_1^{b^3+1} \frac{du}{u}$$

$u = x^3+1$
$du = 3x^2\,dx$
$x=0 \Leftrightarrow u = 0^3+1 = 1$
$x = b \Leftrightarrow u = b^3+1$

$$= \frac{1}{3}\Big[\ln u\Big]_1^{b^3+1}$$

$$= \frac{1}{3}\left[\ln(b^3+1) - \cancel{\ln 1}^{\,0}\right]$$

$$= \frac{1}{3}\ln(b^3+1)$$

$$\lim_{b\to\infty} \frac{1}{3}\ln(b^3+1) = \infty \qquad \boxed{\text{DIVERGENT}}$$