7.3 Optimizing functions of Several Variables

The D-Test: (this plays the role of the 2nd derivative test)

Defn: For $f(x,y)$

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

Remark: D represents the "product" of concavities.

DTest: If (a,b) is a critical point of f, and calculate D at $(x,y)=(a,b)$.

	$f_{xx}>0$	$f_{xx}<0$ (or use f_{yy})
$D>0$	rel. min at (a,b)	rel. max at (a,b)
$D<0$	saddle point at (a,b)	
$D=0$	¿Quién sabe?	

6) Find the relative extreme values of f:

$$f(x,y) = 2xy - 2x^2 - 3y^2 + 4x - 12y + 5$$

$$0 = f_x(x,y) = 2y - 4x + 4$$

$$0 = f_y(x,y) = 2x - 6y - 12$$

Solve [see 10/1 notes] critical point: $(0,-2)$

Q: Is this the location of a rel. max, a rel. min. or saddle point?

$f_{xx}(x,y) = -4$

$f_{xy}(x,y) = 2 = f_{yx}(x,y)$

$f_{yy}(x,y) = -6$

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} -4 & 2 \\ 2 & -6 \end{vmatrix}$$

$$= (-4)(-6) - (2)(2)$$
$$= 24 - 4 = 20 > 0$$

At $(x,y) = (0,-2)$ what is $f_{xx}(0,-2)$?

$f_{xx}(0,-2) = -4 < 0$ [so $f_{yy} < 0$ also]

∴ The graph of f at (0,−2) is "concave <u>down</u>" in both directions.

There is <u>relative</u> <u>maximum</u> at (0, −2).

Now, the <u>value</u> of this relative max is

$$f(0,-2) = 2(0)(-2) - 2\cdot 0^2 - 3\cdot(-2)^2 + 4(0) - 12(-2) + 5$$
$$= 0 - 0 - 12 + 0 + 24 + 5$$
$$= \boxed{17} \text{ at } (x,y) = (0,-2)$$

16) $f(x,y) = x^3 - y^2 - 3x + 6y$

Find critical pts:

$0 = f_x(x,y) = 3x^2 - 3 = 3(x-1)(x+1)$

$0 = f_y(x,y) = -2y + 6 = -2(y-3)$

Solutions: (hence critical points) $(x,y) = (1, 3)$ and $(x,y) = (-1, 3)$

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 6x & 0 \\ 0 & -2 \end{vmatrix} = (6x)(-2) - (0)(0) = -12x$$

At $(x,y) = (1,3)$, $D = -12(1) = -12 < 0$

$\therefore$ saddle point at $(1,3)$.

At $(x,y) = (-1,3)$, $D = -12(-1) = 12 > 0$

But $f_{xx}(-1,3) = 6(-1) = -6$ [so concave down in two directions]

$\therefore$ $f(-1,3)$ is a relative maximum

That is $f(-1,3) = (-1)^3 - (3)^2 - 3(-1) + 6(3)$

$= -1 - 9 + 3 + 18$ and this is the relative maximum at $(-1,3)$

$= 11$

§7.3

22) A company makes two products.

p499 The price function for product A is $p = 16 - x$ thousand dollars $(0 \le x \le 16)$. The price function for product B is $q = 19 - \frac{1}{2}y$ thousand dollars $(0 \le y \le 38)$.

$C(x,y) = 10x + 12y - xy + 6$ thousand dollars

what is $R(x,y) = xp + yq$

$$= x(16-x) + y(19 - \tfrac{1}{2}y)$$

$$\therefore \; P(x,y) = R(x,y) - C(x,y)$$

$$P(x,y) = 16x - x^2 + 19y - \tfrac{1}{2}y^2 - 10x - 12y + xy - 6$$

$$P(x,y) = -x^2 - \tfrac{1}{2}y^2 + 6x + 7y + xy - 6$$

Now maximize $P(x,y)$:

$$0 = P_x(x,y) = -2x + y + 6$$

$$0 = P_y(x,y) = -y + x + 7$$

1st: $2x - y = 6$

2nd: $-x + y = 7$

$\boxed{x = 13}$ units of product A

Sub into 2nd: $-13 + y = 7 \Rightarrow \boxed{y = 20}$ units of product B

$p = 16 - 13 = \boxed{3 \text{ thousand dollars/unit}}$

$q = 19 - \frac{1}{2}(20) = \boxed{9 \text{ thousand dollars/unit}}$

The maximized profit is $P(13,20) = -13^2 - \frac{1}{2}\cdot 20^2 + 6(13) + 7(20) + (13)(20) - 6$

$= -169 - 200 + 78 + 140 + 260 - 6 = \boxed{\$103 \text{ thousand of profit}}$