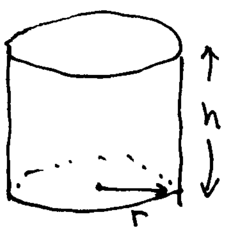
7.5 Lagrange multipliers (cont'd)

#28) Minimize the (area of) material in a open-top cylindrical tank (units: ft^2)

[See Ex #2]

Fixed volume = 120 ft^3



r = radius (ft)

h = height (ft)

Volume of cylinder $= \pi r^2 h = 120 \text{ ft}^3$ = constraint

Area = (area of base) + (area of side)

$= \pi r^2 + 2\pi r h$ = objective function (to be minimized)

$$f(h,r) = \pi r^2 + 2\pi r h$$

$$g(h,r) = \pi r^2 h - 120 = 0$$

$$F(h,r,\lambda) = f(h,r) + \lambda g(h,r)$$
$$= \pi r^2 + 2\pi r h + \lambda(\pi r^2 h - 120)$$

Now find critical point(s) of F:

$$\begin{cases} (1) & 0 = F_h = 2\pi r + \lambda \pi r^2 \\ (2) & 0 = F_r = 2\pi r + 2\pi h + 2\lambda \pi r h \\ (3) & 0 = F_\lambda = \pi r^2 h - 120 \end{cases}$$

Solve (1) and (2) for λ and set them equal to each other (to elimate λ)

$$(1): \quad \frac{-2\cancel{\pi r}}{\cancel{\pi r}^2} = \frac{\lambda \cancel{\pi r^2}}{\cancel{\pi r^2}} \Rightarrow \lambda = \frac{-2}{r}$$

$$(2): \quad \frac{-\cancel{2\pi r}}{\cancel{2\pi r}h} - \frac{\cancel{2\pi h}}{\cancel{2\pi h}r} = \frac{\cancel{2}\lambda \cancel{\pi r h}}{\cancel{2\pi r h}} \Rightarrow \lambda = \frac{-1}{h} - \frac{1}{r}$$

$[\lambda =]\ -\frac{2}{r} = \frac{-1}{h} - \frac{1}{r}$

$-\frac{2}{r} + \frac{1}{r} = \frac{-1}{h}$

$-\frac{1}{r} = \frac{-1}{h} \Rightarrow -r = -h \Rightarrow r = h$

Plug into (3): $\pi r^2 h = 120$

So $\pi r^2 \cdot r = 120$

$\pi r^3 = 120$

$r^3 = \frac{120}{\pi} \Rightarrow$

$$\boxed{\begin{aligned} r &= \sqrt[3]{\frac{120}{\pi}} \approx 3.37 \text{ ft} \\ h &= \sqrt[3]{\frac{120}{\pi}} = 3.37 \text{ ft} \end{aligned}}$$

are the dimension of the cylinder of volume 120 ft^3 which minimizes the area.

7.6 Total Differentials and Approximate Changes

Notation: For a function $f(x,y)$ the total differential, called df, is

$$df = f_x(x,y) \cdot dx + f_y(x,y) \cdot dy$$

or

$$df = \frac{\partial f}{\partial x} \cdot dx + \frac{\partial f}{\partial y} \cdot dy$$

(linearized) net change in elevation

slope in in east-west direction

steps taken in east/west direction

slope in north-south direction

steps taken in north-south direction

10) $z = x^2 \ln y$

$$dz = \frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} \cdot dy$$

$$dz = 2x \ln y \, dx + \frac{x^2}{y} dy$$

14) $w = 3x - xy^{-1} + y^3$

$$dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy \cancel{+ \frac{\partial w}{\partial z} dz}$$

$$dw = (3 - y^{-1}) dx + (xy^{-2} + 3y^2) dy$$

16) $f(x,y,z) = xy + yz + xz$

$$df = f_x(x,y,z)\, dx + f_y(x,y,z)\, dy + f_z(x,y,z)\, dz$$

$$df = (y+z)\, dx + (x+z)\, dy + (y+x)\, dz$$

<u>Use of df in</u> <u>Approximation</u>

22) $f(x,y) = x^3 + xy + y^3$.

Suppose that $(x,y) = (5,3)$ and $(dx, dy) = (\Delta x, \Delta y)$
$= (0.01, -0.01)$

b) For these values, find df.

$$df = f_x \cdot dx + f_y \cdot dy = (3x^2 + y)\, dx + (x + 3y^2) \cdot dy$$

$$= (3 \cdot 5^2 + 3)(0.01) + (5 + 3 \cdot 3^2) \cdot (-0.01)$$

$$= (78)(0.01) + (32)(-0.01) = .78 - .32$$

$$= \boxed{0.46}$$

For these values find

a) $\Delta f = f(x+\Delta x,\ y+\Delta x) - f(x,y)$

[Notes
Finished after class ended:]

$(x,y) = (5,3) \qquad (x+\Delta x,\ y+\Delta y) = (5+.01,\ 3+(-.01))$

$= (5.01,\ 2.99)$

so $\Delta f = f(5.01,\ 2.99) - f(5,3)$

$= [5.01^3 + (5.01)(2.99) + 2.99^3]$

$- [5^3 + (5)(3) + 3^3]$

$= 167.4623 - 167$

$= \boxed{0.4623}$

<u>Remark</u>: Note that $df = 0.46$ is a fairly good approximation to $\Delta f = 0.4623$, but that to calculate Δf, one needs a calculator, while to calculate df, one only needs calculus.