Chap 9 : Differential Equations

what is a differential equation?

ex: [An algebraic equation]

$$x^2 - 8x + 15 = 0$$

Here x is a placeholder for a number.

What does it mean to "solve" this equation?

Solution set $= \{3, 5\}$

To solve the equation means to find all numbers which satisfy the equation:

$$3^2 - 8\cdot 3 + 15 = 0 \quad \text{and} \quad 5^2 - 8\cdot 5 + 15 = 0$$

ex: [A differential equation]

$$\frac{d^2y}{dx^2} + y = 0 \quad \text{or} \quad y'' + y = 0$$

the constant zero function

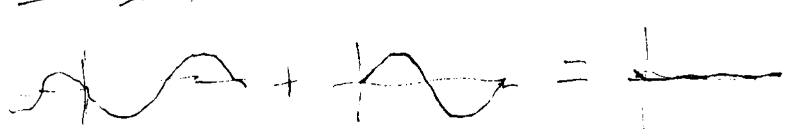
Here y is a placeholder for a function (of x)

To solve this differential equation means to find a function $y = f(x)$ which satisfies the equation.

Like $y = \sin x$. Is this a solution?

$$\frac{dy}{dx} = \cos x, \quad \text{and} \quad \frac{d^2y}{dx^2} = -\sin x$$

So $$\frac{d^2y}{dx^2} + y = -\sin x + \sin x = 0$$



9.1 Separation of Variables

<u>Verifying solutions</u>: If you have a diff. eq. and $y=f(x)$ which is a <u>potential</u> solution, how do you check it?

<u>ex</u>: $\frac{dy}{dx}+3y=0$ is an example of a "first order" diff. eq. that is, the highest order of derivative which appears is <u>first</u> order ($\frac{dy}{dx}$ or y')

a) Is $y=7e^{-3x}$ a solution?

b) Is $y=7e^{-2x}$ a solution?

a) If $y=7e^{-3x}$ then $\frac{dy}{dx}=7e^{-3x}\cdot\underbrace{\frac{d}{dx}[-3x]}_{-3}=-21e^{-3x}$

Then $\frac{dy}{dx}+3y=-21e^{-3x}+3\left(7e^{-3x}\right)$

$=-21e^{-3x}+21e^{-3x}=0$ ← The zero <u>function</u>.

Answer: $\boxed{\text{Yes}}$

b) How about $y=7e^{-2x}$? $\frac{dy}{dx}=7e^{-2x}\cdot\underbrace{\frac{d}{dx}[-2x]}_{-2}=-14e^{-2x}$

Then $\frac{dy}{dx}+3y=-14e^{-2x}+3\left(7e^{-2x}\right)$

$=-14e^{-2x}+21e^{-2x}=7e^{-2x}\neq 0$

Answer: $\boxed{\text{No}}$

"Practice problem 1": For $y'' - 2y' - 3y = 0$

(an example of a second order diff. eqn.),

is $y = e^{-x} + e^{3x}$ a solution?

$$y' = -e^{-x} + 3e^{3x}$$

$$y'' = e^{-x} + 9e^{3x}$$

So
$$\begin{aligned} y'' - 2y' - 3y &= e^{-x} + 9e^{3x} \\ &\quad -2(-e^{-x} + 3e^{3x}) \\ &\quad -3(e^{-x} + e^{3x}) \\ &= e^{-x} + 9e^{3x} \\ &\quad +2e^{-x} - 6e^{3x} \\ &\quad -3e^{-x} - 3e^{3x} \\ &= 0e^{-x} + 0e^{3x} = 0 \quad \checkmark \end{aligned}$$

How do we come up with potential solutions?

ex [A Diff Equ of form $y' = f(x)$]

$$y' = \frac{dy}{dx} = \frac{1}{2}x$$

Answer: $y = \int \frac{1}{2}x\,dx = \frac{1}{4}x^2 + C$ ← there are infinitely many answers.

ex: $y' = 3x^2 + \frac{1}{x} + e^{3x} + \cos x$

$$y = \int \left(3x^2 + \frac{1}{x} + e^{3x} + \cos x\right) dx$$

$$= x^3 + \ln|x| + \frac{1}{3}e^{3x} + \sin x + C$$

Separation of variables

ex: $\frac{dy}{dx} = 3y$

$$dy = 3y\, dx$$

$$\frac{dy}{y} = 3\, dx$$

$$\int \frac{dy}{y} = \int 3\, dx$$

$$\ln|y| + C_2 = 3x + C_1$$

← Remark: We don't need two arbitrary constants!

$$\ln|y| = 3x + \underbrace{(C_1 - C_2)}_{\text{call this } C_3}$$

$$\ln|y| = 3x + C_3$$

Can we solve for y?

$$e^{\ln|y|} = e^{3x + C_3}$$

$$|y| = e^{C_3} \cdot e^{3x}$$

$$y = \pm e^{C_3} \cdot e^{3x}$$

Let $C = \pm e^{C_3}$.

$$\boxed{y = Ce^{3x}}$$

Check: Is $y = Ce^{3x}$ really a solution?

$$\frac{dy}{dx} = 3Ce^{3x} \quad \text{so}$$

$$\frac{dy}{dx} = 3Ce^{3x} = 3\left(Ce^{3x}\right) = 3y \quad \checkmark$$

6) $y^4 y' = 8x$

$$y^4 \frac{dy}{dx} = 8x$$

$$\int y^4 \, dy = \int 8x \, dx$$

$$\frac{1}{5}y^5 = 4x^2 + C_1$$

$$y^5 = 20x^2 + 5C_1$$

$$y = \sqrt[5]{20x^2 + 5C_1}$$

for cosmetic reasons, Let $C = 5C_1$.

$$y = \sqrt[5]{20x^2 + C}$$

Remark: You must include the "C" the moment you integrate, no later.

40) [An "initial value problem"]

$$\begin{cases} y' = y^2 \\ y(2) = -1 \end{cases}$$ ← This will determine a particular value of C. (An "initial condition")

$$\frac{dy}{dx} = y^2$$

$$\int \frac{dy}{y^2} = \int dx$$

$$\int y^{-2}\,dy = \int dx$$

$$-y^{-1} = x + C$$

$$\frac{-1}{y} = x + C$$

What is C? When $x = 2$, $y = -1$:

$$\frac{-1}{(-1)} = (2) + C$$

$$1 = 2 + C \Rightarrow C = -1$$

$$\frac{-1}{y} = x - 1$$

$$\frac{1}{y} = -(x-1) = 1 - x$$

$$\boxed{y = \frac{1}{1-x}}$$

To check would mean to verify

1. $\frac{dy}{dx} = y^2$ AND
2. $y(2) = -1$ (initial condition)

To do 2: $y(2) = \frac{1}{1-2} = \frac{1}{-1} = -1$ ✓