9.2 Three Models of Growth

Unlimited growth

$$\frac{dy}{dt} = ay$$

- $\frac{dy}{dt}$: growth rate of a population (or balance)
- $=$: is proportional to
- y: the population (or balance)
- (a is the constant of proportionality)

$$\int \frac{dy}{y} = \int a\,dt$$

$$\ln|y| = at + C_1$$

$$|y| = e^{at+C_1} = e^{C_1} \cdot e^{at}$$

$$y = \pm e^{C_1} \cdot e^{at} \qquad \text{Let } c = \pm e^{C_1}$$

$$\boxed{y = ce^{at}}$$

What is e? If $t=0$

then $y = ce^0 = c$ so $c = y(0)$.

8) $y' = 6y$

Solution: $y(t) = ce^{6t}$

Limited Growth

$$\frac{dy}{dt} = a(M - y)$$

- $\frac{dy}{dt}$: rate of growth
- a: constant of proportionality
- $(M - y)$: distance below some upper bound M

$$\int \frac{dy}{M-y} = \int a\,dt$$

$$\int \frac{1}{-y+M}\,dy = at + C_1$$

$$\frac{1}{-1}\ln|-y+M| = at + C_1$$

$$\ln|M-y| = -at - C_1$$

$$e^{\ln|M-y|} = e^{-at-C_1} = e^{-C_1}\cdot e^{-at}$$

$$|M-y| = e^{-C_1}\cdot e^{-at}$$

$$M-y = \pm e^{-C_1}\cdot e^{-at}$$

Let $c = \pm e^{-C_1}$

$$M-y = ce^{-at}$$

Now, multiply both sides by -1:

$$y-M = -ce^{-at}$$

$$y = M - ce^{-at}$$

If, in addition, we assume that $y(0) = 0$,

then $0 = M - ce^{0}$

$0 = M - c \Rightarrow c = M$

$y = M - Me^{-at}$ or $\boxed{y = M(1 - e^{-at})}$

20) $y' = 48(2-y)$ and $y(0) = 0$

Note: we must recognize that this diff. eqn. has the form $y' = a(M-y)$ where $a = 48$ and $M = 2$, that is, a limited growth model,

Solution: $y = M(1-e^{-at})$ or

$$y(t) = 2(1-e^{-48t})$$

38) $y' = a(M-y)$ where t = time (weeks)

$y(t)$ = number of people who have heard of the product.

$M = 1{,}000{,}000$

$y' = a(1{,}000{,}000 - y)$

$y(t) = M(1-e^{-at})$

So $y(t) = 1{,}000{,}000(1-e^{-at})$

Other info:

$y(8) = 250{,}000$

Use this to find a,

$250{,}000 = 1{,}000{,}000(1-e^{-a\cdot 8})$

$\frac{250{,}000}{1{,}000{,}000} = \frac{1}{4} = 1 - e^{-8a}$

$\frac{1}{4} - 1 = -\frac{3}{4} = -e^{-8a} \Rightarrow \frac{3}{4} = e^{-8a}$

$$\ln\left(\frac{3}{4}\right) = \ln e^{-8a} = -8a$$

$$a = \frac{\ln(3/4)}{-8} = 0.03596$$

$$\boxed{y(t) = 1{,}000{,}000\left(1 - e^{-0.03596\,t}\right)}$$

$$y(20) = 1{,}000{,}000\left(1 - e^{-0.03596(20)}\right)$$

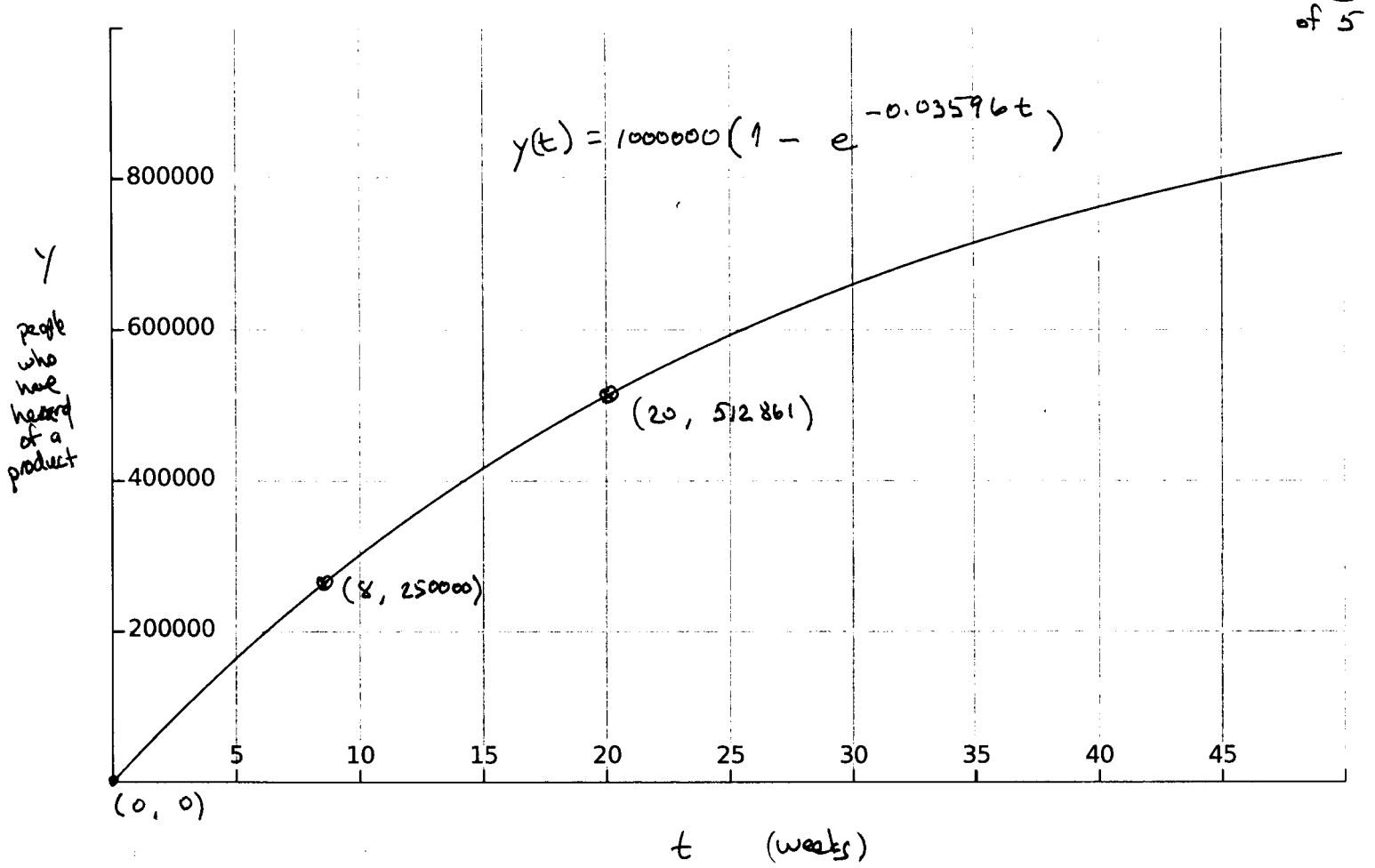
$$= 1{,}000{,}000\left(1 - e^{-0.719205}\right)$$

$$= 1{,}000{,}000\,(1 - 0.487139)$$

$$= 1{,}000{,}000\,(0.512861)$$

$$= 512{,}861$$ people have heard of the product after 20 weeks.

↓ Added after class ended


$y(t) = 1000000\left(1 - e^{-0.03596t}\right)$
y
people who have heard of a product
800000
600000
400000
200000
5
10
15
20
25
30
35
40
45
(0, 0)
(8, 250000)
(20, 512861)
t (weeks)