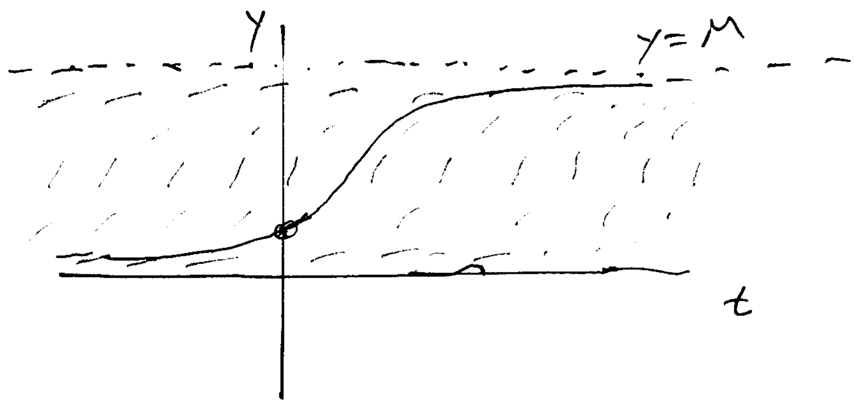
9.2 (cont'd) Three Models of Growth

Logistic Growth

$$\frac{dy}{dt} = a \; y \; (M - y)$$

rate of growth — is — proportional to — present size — upper limit minus present size

Slope field



algebraic Fact:

$$\frac{1}{y(M-y)} = \frac{1}{M}\left[\frac{1}{y} + \frac{1}{M-y}\right]$$

Derivation:

$$\frac{dy}{dt} = a y (M - y)$$

$$\int \frac{dy}{y(M-y)} = \int a\, dt$$

$$\frac{1}{M}\int\left(\frac{1}{y} + \frac{1}{M-y}\right)dy = \int a\, dt$$

$$\frac{1}{M}\Big[\ln|y| - \ln|M-y|\Big] = at + C_1$$

$$\ln|y| - \ln|M-y| = aMt + \underbrace{MC_1}_{C_2}$$

$$\ln\left|\frac{y}{M-y}\right| = aMt + C_2$$

$$e^{\ln\left|\frac{y}{M-y}\right|} = e^{aMt + C_2}$$

$$\left|\frac{y}{M-y}\right| = e^{C_2} \cdot e^{aMt}$$

$$\frac{y}{M-y} = \pm e^{C_2} \cdot e^{aMt}$$

Take reciprocals:

$$\frac{M-y}{y} = \underbrace{\pm e^{-C_2}}_{c} \cdot e^{-aMt}$$

$$\frac{M}{y} - \frac{y}{y} = c\, e^{-aMt}$$

$$\frac{M}{y} - 1 = c\, e^{-aMt}$$

$$\frac{M}{y} = 1 + c\, e^{-aMt}$$

Take reciprocals:

$$\frac{y}{M} = \frac{1}{1 + c e^{-aMt}}$$

$$\boxed{y(t) = \frac{M}{1 + c e^{-aMt}}}$$

Now, what is c? Set $t=0$.

$$y(0) = \frac{M}{1+c}$$

Solve for c:

$$c = \frac{M - y(0)}{y(0)}$$

9.2 34) $y' = 3y - 6y^2$, $y(0) = \frac{1}{6}$

Write as $\frac{dy}{dt} = 6y\left(\frac{1}{2} - y\right)$ to have the form
$y' = ay(M-y)$

so $a = 6$ and $M = \frac{1}{2}$ and $aM = 3$.

Solution: $y(t) = \frac{M}{1+ce^{-aMt}} = \frac{\frac{1}{2}}{1+ce^{-3t}}$

What is c?

$\frac{1}{6} = y(0) = \frac{\frac{1}{2}}{1+c\cdot e^{0}} = \frac{\frac{1}{2}}{1+c} \cdot \frac{2}{2}$

$\frac{1}{6} = \frac{1}{2+2c} \Rightarrow 6 = 2+2c$
$4 = 2c$
$2 = c$

$y(t) = \frac{\frac{1}{2}}{1+2e^{-3t}} \cdot \frac{2}{2}$

$$\boxed{y(t) = \frac{1}{2+4e^{-3t}}}$$

9.2 43) A Rumor is spreading; so logistic growth.

$$y' = ay(M-y)$$

So solution:
$$y(t) = \frac{M}{1+ce^{-aMt}}$$

we need to determine a, M and c. Easy: $M=800$

because there are 800 people in the airport to potentially hear the rumor.

$$y(t) = \frac{800}{1+ce^{-800at}} \qquad \text{Still need } a, c.$$

Info yet to use: t = time (minutes)

Given: $y(0) = 1$ person believes the rumor

also: $y(10) = 200$ people " " "

Idea for rest of problem:

$t=0$: $$1 = \frac{800}{1+ce^{-800a\cdot 0}}$$

$t=10$: $$200 = \frac{800}{1+ce^{-800a\cdot(10)}}$$

This is a (nonlinear) system of two equations, two variables.

[The ugly details will follow in notes]

Intro to 9.3 <u>First order linear differential equations</u>

has the form

$$\boxed{y' + p(x)\cdot y = q(x)}$$

<u>ex</u>: $y' + 2xy = x^3$ so $p(x) = 2x$, $q(x) = x^3$

<u>ex</u>: (i) $y' - 2xy = 0$ $p(x) = -2x$, $q(x) = 0$

(ii) $(x+1)\,y' + y = 2x$ Not in standard form

$$y' + \frac{1}{x+1}\cdot y = \frac{2x}{x+1}$$

<u>Now</u> in std. form

$$p(x) = \frac{1}{x+1}$$

$$q(x) = \frac{2x}{x+1}$$

We can <u>always</u> solve this type of diff. eqn.