From last time:

9.2 43) (continued)

$$y(t) = \frac{M}{1+ce^{-aMt}}$$ where $\boxed{M = 800}$

Also given: $\begin{cases} y(0) = 1 \\ y(10) = 200 \end{cases}$

$$y(t) = \frac{800}{1+ce^{-800at}}$$ And when $t=0$, $y=1$

$$1 = \frac{800}{1+c}$$ because $e^0 = 1$.

$$1+c = 800$$
$$\boxed{c = 799}$$ What's left? we need a.

$$y(t) = \frac{800}{1+799e^{-800at}}$$ And when $t=10$, $y=200$

$$200 = \frac{800}{1+799e^{-8000a}}$$ because $-800t = -800(10) = -8000$

$$\frac{1}{4} = \frac{200}{800} = \frac{1}{1+799e^{-8000a}}$$

$$4 = 1+799e^{-8000a}$$ Subtract 1:

$$3 = 799\, e^{-8000a}$$

$$\frac{3}{799} = e^{-8000a}$$

$$\ln\left(\frac{3}{799}\right) = \ln e^{-8000a} = -8000a$$

$$\frac{1}{-8000}\ln\left(\frac{3}{799}\right) = a \qquad \text{so}$$

$$a = 6.981 \cdot 10^{-4} = 0.0006981$$

$$\text{so} \quad aM = \cancel{5.585} = 0.0006981 \cdot 800 = 0.5585$$

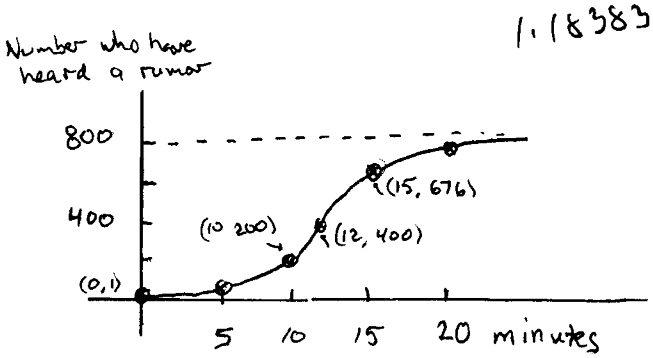
$$y(t) = \frac{M}{1 + 799e^{-aMt}} = \boxed{\frac{800}{1 + 799e^{-0.5585\,t}}}$$

What is $y(15)$?

$$y(15) = \frac{800}{1 + 799\, e^{-0.5585(15)}} = \frac{800}{1 + 799\, e^{-8.377}}$$

$$= \frac{800}{1 + 799\,(0.00023)} = \frac{800}{1 + .18383}$$

$$= \frac{800}{1.18383} = \boxed{676}$$



9.3 First ORder Linear Differential Equations

Idea: Using the product rule backwards.

ex (forwards): If $y = f(x)$ for some f

$$\frac{d}{dx}[x^3 y] = 3x^2 \cdot y + x^3 \cdot y'$$

Backwards: $$x^3 y' + 3x^2 y = \frac{d}{dx}[x^3 y]$$

ex: $$y' + \frac{4}{x} y = 2x$$

$p(x) = \frac{4}{x}$

$q(x) = 2x$

First scratch work

Find the integrating factor $\boxed{I(x) = e^{\int p(x)\,dx}}$

Step 1 $P(x) = \frac{4}{x}$

Step 2 $$\int p(x)\,dx = \int \frac{4}{x}\,dx = 4\int \frac{dx}{x} = 4\ln|x| = \ln|x|^4 = \ln x^4$$

Remark: No "+C" needed.

Step 3 $$I(x) = e^{\int p(x)dx} = e^{\ln x^4} = x^4$$

Now multiply the standard form of the Linear Diff Eq by $I(x) = x^4$

$$x^4 y' + x^4 \cdot \frac{4}{x} y = x^4 \cdot 2x$$

$$x^4 y' + 4x^3 y = 2x^5$$

Product Rule Backwards: $\frac{d}{dx}\left[x^4 y\right] = 2x^5$

$\frac{d}{dx}\left[I(x) \cdot y\right]$

$$\int \frac{d}{dx}\left[x^4 y\right] dx = \int 2x^5 \, dx$$

we need the "+C"

$$x^4 y = \frac{1}{3} x^6 + C$$

multiply by x^{-4}: $x^{-4} \cdot x^4 y = \frac{1}{3} x^6 \cdot x^{-4} + C x^{-4}$

$$y = \frac{1}{3} x^2 + C x^{-4} = \frac{1}{3} x^2 + \frac{C}{x^4}$$

8) $xy' - y = x$

Step 0 Put into standard form: multiply by $\frac{1}{x}$:

$$\frac{1}{x} x y' - \frac{1}{x} y = \frac{1}{x} \cdot x$$

$$y' - \frac{1}{x} y = 1$$

scratch work

Find $I(x)$:

$$P(x) = \frac{-1}{x}$$

$$\int P(x) dx = \int \frac{-1}{x} dx = -\ln x = \ln x^{-1}$$

$$I(x) = e^{\int P(x) dx} = e^{\ln x^{-1}} = x^{-1} = \frac{1}{x}$$

Multiply the standard form by $I(x) = \frac{1}{x}$

$$\frac{1}{x} y' - \frac{1}{x^2} y = \frac{1}{x}$$

$$x^{-1} y' - x^{-2} y = x^{-1}$$

$$\frac{d}{dx}[x^{-1} y] = x^{-1}$$

$$x^{-1} y = \int x^{-1} dx$$

$$x^{-1} y = \ln x + C \qquad \text{Multiply by } x:$$

$$y = x \ln x + C x$$

22) $y' + 4y = e^{-3x}$, $y(0) = 4$

$p(x) = 4$

$\int p(x)\,dx = \int 4\,dx = 4x$

$I(x) = e^{\int p(x)dx} = e^{4x}$

$$e^{4x} y' + 4e^{4x} y = e^{-3x} \cdot e^{4x} = e^x$$

$$\frac{d}{dx}\left[e^{4x} y\right] = e^x$$

$$e^{4x} y = \int e^x dx$$

$$e^{4x} y = e^x + C$$

when $x=0$, $y=4$:

$$e^0 \cdot y = e^0 \cdot 4 = e^0 + C$$

$$4 = 1 + C \Rightarrow C = 3$$

$$e^{4x} y = e^x + 3$$

Multiply by e^{-4x}:

$$\boxed{y = e^{-3x} + 3e^{-4x}}$$

because $e^x \cdot e^{-4x} = e^{x+(-4x)} = e^{-3x}$

46) You deposit \$8000 into a bank acct paying 5% interest compounded continuously. You withdraw at the rate of \$1000 per year. If $y(t)$ is your balance

$$\begin{cases} y' = 0.05y - 1000 \\ y(0) = 8000 \end{cases}$$

$$y' - 0.05y = -1000$$

$$p(t) = -0.05$$

$$\int p\,dt = \int -0.05\,dt = -0.05t$$

$$I(x) = e^{\int p\,dt} = e^{-0.05t}$$

$$e^{-0.05t}y' - 0.05e^{-0.05t} = -1000e^{-.05t}$$

$$\frac{d}{dt}\left[e^{-0.05t}y\right] = -1000e^{-0.05t}$$

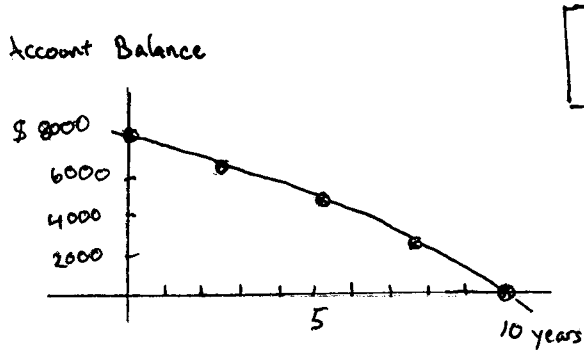
$$e^{-0.05t}y = \frac{-1000}{-0.05}e^{-0.05t} + C$$

$$e^{-0.05t}y = 20000e^{-0.05t} + C$$

multiply by $e^{0.05t}$: $y(t) = 20{,}000 + Ce^{0.05t}$

Use that when $t=0$, $y=8000$: $8000 = y(0) = 20{,}000 + C \Rightarrow C = -12{,}000$

$$\boxed{y(t) = 20{,}000 - 12{,}000e^{0.05t}}$$



Remark: You run out of money after 10.2165 years.