Warmup

ex: where did the formula for $\int \tan x\,dx$ come from?

$$\int \tan x\,dx = \int \frac{\sin x\,dx}{\cos x} = \int \frac{-d(\cos x)}{\cos x} = -\ln|\cos x| + C$$

$$= \ln|\cos x|^{-1} + C = \ln|\sec x| + C$$

$$\int \cot dx = \int \frac{\cos x\,dx}{\sin x} = \int \frac{d(\sin x)}{\sin x} = \ln|\sin x| + C$$

$$= -\ln|\sin x|^{-1} + C = -\ln|\csc x| + C$$

ex: $\int 2^x\,dx = ?$ NOTE: $2^x = e^{\ln 2^x} = e^{x\ln 2} = e^{(\ln 2)x}$

$$\int 2^x\,dx = \int e^{(\ln 2)x}\,dx$$ Now: $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C$

$$= \cancel{\frac{1}{\ln 2}\int e} = \frac{1}{\ln 2}e^{(\ln 2)x} + C = \frac{1}{\ln 2}2^x + C$$

ex: $\int \frac{1}{x^2+1}\,dx = \arctan x + C = \tan^{-1} x + C$

Start with: $\tan(\arctan x) = x$

$$\frac{d}{dx}\left[\tan(\arctan x)\right] = \frac{d}{dx}[x]$$

$$\sec^2(\arctan x)\cdot\frac{d}{dx}[\arctan x] = 1$$

$$\frac{d}{dx}[\arctan x] = \frac{1}{\sec^2(\arctan x)}$$

But $\sec^2(\arctan x)$ is awkward.

Use: $\sec^2\theta = \tan^2\theta + 1$ with $\theta = \arctan x$, so that

$$\sec^2(\arctan x) = [\sec(\arctan x)]^2 = \{\tan(\arctan x)\}^2 + 1$$
$$= x^2 + 1$$

$\therefore \frac{d}{dx}[\arctan x] = \frac{1}{x^2+1}$ so $\int \frac{1}{x^2+1}\,dx = \tan^{-1}(x) + C$

what about $\int \frac{1}{x^2+a^2}\,dx$?

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a^2}\int \frac{1}{\left(\frac{x}{a}\right)^2+1}\,dx = \frac{1}{a}\int \frac{1}{\left(\frac{x}{a}\right)^2+1}\,d\left(\frac{x}{a}\right)$$

$$= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

7.1 Integration by parts

Remark: (1) Integration by substitution = chain rule + antiderivative

(2) Integration by parts = product rule + antiderivative.

Main idea:
$$\frac{d}{dx}[f(x)\cdot g(x)] = f'(x)\cdot g(x) + f(x)\cdot g'(x)$$
$$\int \frac{d}{dx}[f(x)\cdot g(x)]\,dx = \int g(x)\cdot f'(x)\,dx + \int f(x)\cdot g'(x)\,dx$$
$$f(x)\cdot g(x) = \int g(x)\,d[f(x)] + \int f(x)\,d[g(x)]$$

Let $\begin{cases} u = f(x) \\ v = g(x) \end{cases}$ Then $uv = \int v\,du + \int u\,dv$

so
$$\boxed{\underbrace{\int u\,dv}_{\text{ugly and hopeless}} = uv - \underbrace{\int v\,du}_{\text{nice (we hope)}}}$$

ex: $\int x\cos x \, dx$

$u =$ part to be differentiated
$dv =$ part to be integrated

$\int u\,dv = uv - \int v\,du$

Try $u = x$ and $dv = \cos x \, dx$

$du = 1dx$ $\quad$ $v = \sin x$ ← No "+ C" is needed.

$$\int x\cos x\,dx = x\sin x - \int \sin x \, dx$$
$$= x\sin x - (-\cos x) + C$$
$$= x\sin x + \cos x + C$$

Check: $\frac{d}{dx}\left[x\sin x + \cos x + C\right]$
$$= 1\cdot\cancel{\sin x} + x\cos x - \cancel{\sin x} + 0 = x\cos x$$

Remark: We could have taken $v = \sin x + 17$.

$$\int x\cos x\,dx = x(\sin x + 17) - \int(\sin x + 17)\,dx$$
$$= x\sin x + 17x + \cos x - 17x + C$$
$$= x\sin x + \cos x + C$$

$\int u\,dv = uv - \int v\,du$ M151 6/16/15

43) $$\int x e^{-2x}\,dx = (x)\left(-\tfrac{1}{2}e^{-2x}\right) - \int\left(-\tfrac{1}{2}e^{-2x}\right)dx$$

$u = x$	$dv = e^{-2x}\,dx$
$du = dx$	$v = -\frac{1}{2}e^{-2x}$

$$= -\tfrac{1}{2}xe^{-2x} + \tfrac{1}{2}\int e^{-2x}\,dx$$

$$= -\tfrac{1}{2}xe^{-2x} + \tfrac{1}{2}\left(-\tfrac{1}{2}e^{-2x}\right) + C$$

$$= -\tfrac{1}{2}xe^{-2x} - \tfrac{1}{4}e^{-2x} + C$$

ex: $$\int \ln x\,dx = \int (\ln x)\cdot 1\,dx = (*)$$

We have no choice but to take:

$u = \ln x$	$dv = 1\,dx$
$du = \frac{1}{x}\,dx$	$v = x$

$$(*) = (\ln x)(x) - \int x\cdot\tfrac{1}{x}\,dx$$

$$= x\ln x - \int dx = x\ln x - x + C$$

ex: $$\int \arctan x\,dx = x\arctan x - \int \frac{x}{x^2+1}\,dx$$

$u = \arctan x$	$dv = dx$
$du = \frac{1}{x^2+1}\,dx$	$v = x$

$$= x\arctan x - \tfrac{1}{2}\int \frac{2x\,dx}{x^2+1}$$

$$= x\arctan x - \tfrac{1}{2}\int \frac{d(x^2+1)}{x^2+1}$$

$$= x\arctan x - \tfrac{1}{2}\ln(x^2+1) + C$$

ex: $\int (x^2+5x)\, e^{3x}\, dx = (x^2+5x)\left(\frac{1}{3}e^{3x}\right) - \int (2x+5)\left(\frac{1}{3}e^{3x}\right)dx$

$$\boxed{\begin{array}{ll} u = x^2+5x & dv = e^{3x}dx \\ du = (2x+5)dx & v = \frac{1}{3}e^{3x} \end{array}}$$

$$\boxed{\begin{array}{ll} u = 2x+5 & dv = \frac{1}{3}e^{3x}dx \\ du = 2\,dx & v = \frac{1}{9}e^{3x} \end{array}}$$

$$= (x^2+5x)\left(\frac{1}{3}e^{3x}\right) - \left[(2x+5)\left(\frac{1}{9}e^{3x}\right) - \int (2)\left(\frac{1}{9}e^{3x}\right)dx\right]$$

$$= (x^2+5x)\left(\frac{1}{3}e^{3x}\right) - (2x+5)\left(\frac{1}{9}e^{3x}\right) + \int (2)\left(\frac{1}{9}e^{3x}\right)dx$$

Yet another integration by parts!

$$\boxed{\begin{array}{ll} u = 2 & dv = \frac{1}{9}e^{3x}dx \\ du = 0\,dx & v = \frac{1}{27}e^{3x} \end{array}}$$

$(0)\left(\frac{1}{27}e^{3x}\right)$

$$= (x^2+5x)\left(\frac{1}{3}e^{3x}\right) - (2x+5)\left(\frac{1}{9}e^{3x}\right) + (2)\left(\frac{1}{27}e^{3x}\right) - \underbrace{\int 0\,dx}_{+C}$$

$$= \left(\frac{1}{3}x^2+\frac{5}{3}x\right)e^{3x} + \left(-\frac{2}{9}x - \frac{5}{9}\right)e^{3x} + \frac{2}{27}e^{3x} + C$$

etc.

Same problem using
Tabular integration by parts

ex $\int (x^2+5x)\, e^{3x}\, dx \quad = \quad (*)$

signs		differentiate	integrate
+	$\longrightarrow$	x^2+5x	e^{3x}
−	$\longrightarrow$	$2x+5$	$\frac{1}{3}e^{3x}$
+	$\longrightarrow$	2	$\frac{1}{9}e^{3x}$
−	$\longrightarrow$	0	$\frac{1}{27}e^{3x}$

$$(*) = (x^2+5x)\left(\tfrac{1}{3}e^{3x}\right) - (2x+5)\left(\tfrac{1}{9}e^{3x}\right) + (2)\left(\tfrac{1}{27}e^{3x}\right) - \int (0)\left(\tfrac{1}{27}e^{3x}\right)dx$$

$$= \left[\frac{x^2+5x}{3} - \frac{2x+5}{9} + \frac{2}{27}\right]e^{3x} + C$$

$$= \left[9(x^2+5x) - 3(2x+5) + 2\right]\frac{e^{3x}}{27} + C$$

$$= (9x^2+45x-6x-15+2)\,\frac{e^{3x}}{27} + C$$

$$= (9x^2+39x-13)\,\frac{e^{3x}}{27} + C$$