7.1 Integration by parts (continued)

<u>Definite</u> integral, by parts

24) $$\int_0^1 (x^2+1)\, e^{-x}\, dx$$

By TI-84 the answer <u>should</u> be 0.792 723 35

<u>Signs</u>		<u>Diff.</u>	<u>Int.</u>
$+$	$\longrightarrow$	x^2+1	e^{-x}
$-$	$\longrightarrow$	$2x$	$-e^{-x}$
$+$	$\longrightarrow$	2	e^{-x}
		0	$-e^{-x}$

$$\text{answer} = (x^2+1)(-e^{-x})\Big|_0^1 - (2x)(e^{-x})\Big|_0^1 + (2)(-e^{-x})\Big|_0^1$$

$$= -(x^2+1)e^{-x}\Big|_0^1 - 2xe^{-x}\Big|_0^1 - 2e^{-x}\Big|_0^1$$

$$= \left[-2e^{-1} + e^0\right] + \left[-2e^{-1} + 0\right] + \left[-2e^{-1} + 2e^0\right]$$

$$= -2e^{-1} + 1 - 2e^{-1} - 2e^{-1} + 2 = \boxed{3 - 6e^{-1}}$$

$$\approx .792\ 723\ 353$$

ex: $\int e^{3x} \sin 5x \, dx$

Strategy: Do two integrations by parts, then use algebra to solve for the integral.

Signs		Diff	Int
$+$	$\longrightarrow$	$\sin 5x$	e^{3x}
$-$	$\longrightarrow$	$5\cos 5x$	$\frac{1}{3}e^{3x}$
$+$	$\longrightarrow$	$-25\sin 5x$	$\frac{1}{9}e^{3x}$

$$\int e^{3x} \sin 5x \, dx = (\sin 5x)\left(\tfrac{1}{3}e^{3x}\right) - (5\cos 5x)\left(\tfrac{1}{9}e^{3x}\right) - \frac{25}{9}\int e^{3x}\sin 5x\,dx$$

$$I(x) = \frac{1}{3}e^{3x}\sin 5x - \frac{5}{9}e^{3x}\cos 5x - \frac{25}{9}I(x)$$

$$9I(x) = 3e^{3x}\sin 5x - 5e^{3x}\cos 5x - 25I(x)$$

$$25I(x) + 9I(x) = 3e^{3x}\sin 5x - 5e^{3x}\cos 5x$$

$$34I(x) =$$

$$I(x) = \int e^{3x}\sin 5x\,dx = \frac{3}{34}e^{3x}\sin 5x - \frac{5}{34}e^{3x}\cos 5x + C$$

More generally: $\displaystyle\int e^{ax}\sin bx\,dx = \frac{ae^{ax}}{a^2+b^2}\sin bx - \frac{b}{a^2+b^2}e^{ax}\cos bx + C$

Remark: In this example we let $u = \sin 5x$ and $dv = e^{3x}$

We could have also let $u = e^{3x}$ and $dv = \sin 5x$.

ex: Show how to derive the reduction formula

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$

$$\int \cos^n x\,dx = \int \cos^{n-1}x \cdot \cos x\,dx$$

Let: $u = \cos^{n-1} x$ $\quad dv = \cos x\,dx$

Sign	Diff	Int
$+$	$\to \cos^{n-1}x$	$\cos x$
$-$	$\to (n-1)\cos^{n-2}x\cdot(-\sin x)$	$\to \sin x$

$$\int \cos^n x\,dx = \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x\sin^2 x\,dx$$

Now: Use $\sin^2 x = 1 - \cos^2 x$ to say

$$\cos^{n-2}x\sin^2 x = \cos^{n-2}x\left(1-\cos^2 x\right) = \cos^{n-2}x - \cos^n x$$

$$\int \cos^n x\,dx = \cos^{n-1}x\sin x + (n-1)\int\cos^{n-2}x\,dx - (n-1)\int \cos^n x\,dx$$

$$I(x) = \cos^{n-1}x\sin x + (n-1)\int\cos^{n-2}x\,dx - (n-1)\,I(x)$$

$$n\,I(x) = I(x) + (n-1)I(x) = \cos^{n-1}x\sin x + (n-1)\int\cos^{n-2}x\,dx$$

$$I(x) = \int\cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int\cos^{n-2}x\,dx$$

Application $(n=2)$: $\displaystyle\int \cos^2 x\,dx = \frac{1}{2}\cos x\sin x + \frac{1}{2}\int 1\,dx$

$$= \frac{1}{2}\cos x\sin x + \frac{1}{2}x + C$$

7.2 Trigonometric Integrals

ex: $\int \sin^5 x \cos x \, dx$ We can do this by substitution.

[see Remark ①]

$u = \sin x$

$du = \cos x \, dx$

$$= \int u^5 \, du = \frac{1}{6} u^6 + C = \frac{1}{6} \sin^6 x + C$$

OR by "in-line substitution".

$$\int \sin^5 x \underbrace{\cos x \, dx}_{d(\sin x)} = \int \sin^5 x \, d(\sin x) \; \left[= \int u^5 du \right]$$

$$\left[= \frac{1}{6} u^6 + C \right] = \frac{1}{6} \sin^6 x + C$$

ex: $\displaystyle\int \frac{\sin x \, dx}{\sqrt{\cos x}} = -\int \frac{-\sin x \, dx}{\sqrt{\cos x}} = -\int \frac{d(\cos x)}{\sqrt{\cos x}}$

[see Remark ②]

$$= -\int \frac{du}{\sqrt{u}} = -\int u^{-1/2} \, du$$

$u = \cos x$

$du = d(\cos x)$

$= -\sin x \, dx$

$$= -2u^{1/2} + C = -2(\cos x)^{1/2} + C$$

Normal way: $\displaystyle -\int \frac{-\sin x \, dx}{\sqrt{\cos x}} = -\int \frac{du}{\sqrt{u}} = -\int u^{-1/2} du$

$$= -2u^{1/2} + C = -2(\cos x)^{1/2} + C$$

Remarks:
1. $d(\sin x) = \cos x \, dx$
2. $d(\cos x) = -\sin x \, dx$
3. $d(\tan x) = \sec^2 x \, dx$
4. $d(\sec x) = \sec x \tan x \, dx$
5. $d(\cot x) = -\csc^2 x \, dx$
6. $d(\csc x) = -\csc x \cot x \, dx$

ex: [See Remark (8)] $\int \tan^8 x \underbrace{\sec^2 x\, dx}_{d(\tan x)} = \int u^8\, du = \frac{u^9}{9} + C$

$= \frac{1}{9}\tan^9 x + C$

$u = \tan x$
$du = \sec^2 x\, dx$

ex: [See Remark (9)] $\int \sec^5 x \tan x\, dx = \int \sec^4 x \underbrace{\sec x \tan x\, dx}_{d(\sec x)}$

$u = \sec x$
$du = \sec x \tan x\, dx$

$= \int u^4\, du = \frac{u^5}{5} + C = \frac{1}{5}\sec^5 x + C$

ex: what if the pattern doesn't quite fit?

$$\int \sin^2 x \cos^3 x\, dx = \int \sin^2 x \cos^2 x \cos x\, dx$$

$$= \int \sin^2 x\,(1 - \sin^2 x) \underbrace{\cos x\, dx}_{d(\sin x)}$$

$u = \sin x$
$du = \cos x\, dx$

$$= \int u^2 (1 - u^2)\, du$$

$$= \int (u^2 - u^4)\, du = \frac{u^3}{3} - \frac{u^5}{5} + C$$

$$= \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$$