7.2 Trig integrals (cont'd)

warmup: Three ways to calculate $\int 2\sin x \cos x \, dx$

(1) $u = \sin x$, $du = \cos x \, dx$

$$\int 2 \sin x \cos x \, dx = \int 2u \, du$$
$$= u^2 + C_1 = \sin^2 x + C_1$$

(2) $u = \cos x$, $du = -\sin x \, dx$

$$= -\int 2\cos x(-\sin x) \, dx = -\int u \, du$$
$$= -u^2 + C_2 = -\cos^2 x + C_2$$

(3) $$\int 2 \sin x \cos x \, dx = \int \sin 2x \, dx$$

$$= \frac{-1}{2} \cos 2x + C_3$$

Q. Are these describing the same function? A: They better be.

(1) vs (2) $$\sin^2 x + C_1 = 1 - \cos^2 x + C_1 = -\cos^2 x + (C_1 + 1)$$
$$= -\cos^2 x + C_2$$

provided we take $C_2 = C_1 + 1$.

(1) vs (3)? $$-\frac{1}{2}\cos 2x + C_3 = -\frac{1}{2}(1 - 2\sin^2 x) + C_3$$

$$= -\frac{1}{2} + \sin^2 x + C_3 = \sin^2 x + \left(C_3 - \frac{1}{2}\right)$$

same as (1) provided that $C_1 = C_3 - \frac{1}{2}$

(2) vs (3)? $$-\frac{1}{2}\cos 2x + C_3 = -\frac{1}{2}(2\cos^2 x - 1) + C_3$$
$$= -\cos^2 x + \left(\frac{1}{2} + C_3\right)$$

same as (2) provided that $C_2 = \frac{1}{2} + C_3$

ex: [Integrand contains an odd power of cos x]

$$\int \sin^6 x \cos^3 x \, dx = \int \sin^6 x \cos^2 x \cos x \, dx$$

$$= \int \sin^6 x (1 - \sin^2 x) \cos x \, dx$$

Now let $u = \sin x$, $du = \cos x \, dx$

$$= \int u^6 (1 - u^2) \, du$$

$$= \int (u^6 - u^8) \, du = \frac{u^7}{7} - \frac{u^9}{9} + C$$

$$= \frac{1}{7} \sin^7 x - \frac{1}{9} \sin^9 x + C$$

Alternatively: $\int u^6 (1-u^2) \, du$ ← By parts

signs		diff	int
+	$\longrightarrow$	$1-u^2$	u^6
−	$\longrightarrow$	$-2u$	$\frac{1}{7} u^7$
+	$\longrightarrow$	-2	$\frac{1}{56} u^8$
		0	$\frac{1}{9 \cdot 8 \cdot 7} u^9$

$$\int u^6 (1-u^2) \, du = (1-u^2)\left(\frac{1}{7} u^7\right) + (2u)\left(\frac{1}{8 \cdot 7} u^8\right) - (2)\left(\frac{1}{9 \cdot 8 \cdot 7} u^9\right) + C$$

$$= (1 - \sin^2 x) \cdot \frac{1}{7} \sin^7 x + 2 \sin x \cdot \frac{1}{56} \sin^8 x - \frac{2}{9 \cdot 8 \cdot 7} \sin^9 x + C$$

ex: [The integrand contains an Odd power of sines]

$$\int \sin^5 x \cos^4 x \, dx = \int \cos^4 x \sin^4 x \sin x \, dx$$

$$= \int \cos^4 x (\sin^2 x)^2 \sin x \, dx$$

$$= -\int \cos^4 x (1-\cos^2 x)^2 (-\sin x) \, dx$$

$u = \cos x$
$du = -\sin x \, dx$

$$= -\int u^4 (1-u^2)^2 \, du$$

$$= -\int u^4 (u^4 - 2u^2 + 1) \, du$$

$$= \int (-u^8 + 2u^6 - u^4) \, du$$

$$= -\frac{u^9}{9} + \frac{2}{7}u^7 - \frac{u^5}{5} + C$$

$$= -\frac{1}{9}\cos^9 x + \frac{2}{7}\cos^7 x - \frac{1}{5}\cos^5 x + C$$

ex: [Even power of both sine and cosine.]

$$\int \sin^2 x \cos^2 x \, dx = \int (\sin x \cos x)^2 \, dx$$

Recall

$\cos^2 x = \frac{1+\cos 2x}{2} = \frac{1}{2} + \frac{1}{2}\cos 2x$

$\sin^2 x = \frac{1-\cos 2x}{2} = \frac{1}{2} - \frac{1}{2}\cos 2x$

$\sin x \cos x = \frac{1}{2}\sin 2x$

$$= \int \left(\frac{1}{2}\sin 2x\right)^2 dx$$

$$= \frac{1}{4}\int \sin^2 2x \, dx$$

$$= \frac{1}{4}\int \left(\frac{1}{2} - \frac{1}{2}\cos 2(2x)\right) dx$$

$$= \int \left(\frac{1}{8} - \frac{1}{8}\cos 4x\right) dx$$

$$= \frac{1}{8}x - \frac{1}{8}\cdot\frac{1}{4}\sin 4x + C$$

$$= \frac{1}{8}x - \frac{1}{32}\sin 4x + C$$

ex: [Even power of secants.]

$$\int \sec^6 x \tan^2 x \, dx = \int \tan^2 x \sec^4 x \underbrace{\sec^2 x \, dx}_{d(\tan x)}$$

$$= \int \tan^2 x \, (\sec^2 x)^2 \sec^2 x \, dx$$

$$= \int \tan^2 x \, (\tan^2 x + 1)^2 \sec^2 x \, dx$$

$u = \tan x$
$du = \sec^2 x \, dx$

$$= \int u^2 (u^2+1)^2 \, du = \int u^2 (u^4 + 2u^2 + 1) \, du$$

$$= \int (u^6 + 2u^4 + u^2) \, du$$

$$= \frac{u^7}{7} + \frac{2}{5} u^5 + \frac{u^3}{3} + C = \frac{1}{7} \tan^7 x + \frac{2}{5} \tan^5 x + \frac{1}{3} \tan^3 x + C$$

ex: [odd power of tangent in an integrand with sec x and tan x.]

$$\int \sec^3 x \tan^3 x \, dx = \int \cancel{\sec^2 x \;\; \sec x \tan x} \, dx$$

$$= \int \sec^2 x \tan^2 x \sec x \tan x \, dx$$

$$= \int \sec^2 x \, (\sec^2 x - 1) \sec x \tan x \, dx$$

$u = \sec x$
$du = \sec x \tan x \, dx$

$$= \int u^2 (u^2 - 1) \, du = \int (u^4 - u^2) \, du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x + C$$

Remark: what if you have a odd power of sec x and an even power of tan x?

I dunno. Good luck.

Squirelly special cases

TRICK

ex: $\int \sec x \, dx = \int \frac{\sec x + \tan x}{\sec x + \tan x} \cdot \sec x \, dx$

$= \int \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} \, dx$

$u = \sec x + \tan x$

$du = \sec x \tan x + \sec^2 x$

$= \int \frac{du}{u} = \ln|u| + C$

$= \ln|\sec x + \tan x| + C$

ex: $\int \csc x \, dx = -\ln|\csc x + \cot x| + C$ ← Exercise:

Textbook: $\ln|\csc x - \cot x| + C$ ↙ Show these are equal.

ex: $\int \sec^3 x \, dx = \int \sec x \sec^2 x \, dx$ By Parts.

$u = \sec x \qquad dv = \sec^2 x \, dx$

$du = \sec x \tan x \, dx \qquad v = \tan x$

$= \sec x \tan x - \int \sec x \tan^2 x \, dx$

$= \sec x \tan x - \int \sec x (\sec^2 x - 1) dx$

$\int \sec^3 x \, dx = \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$

$= \sec x \tan x - \int \sec^3 x \, dx + \ln|\sec x + \tan x|$

$2\int \sec^3 x \, dx = \sec x \tan x + \ln|\sec x + \tan x|$

$\int \sec^3 dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C$