Loose end in Section 7.2

ex: $\int \sin 5x \sin 3x \, dx$

You're on a two-hour flight, no integral tables or trig identities on hand. How do you do this?

$$\cos 5x \cos 3x + \sin 5x \sin 3x = \cos(5x-3x) = \cos 2x$$

subtract:

$$\cos 5x \cos 3x - \sin 5x \sin 3x = \cos(5x+3x) = \cos 8x$$

$$2 \sin 5x \sin 3x = \cos 2x - \cos 8x$$

So

$$\sin 5x \sin 3x = \tfrac{1}{2}\cos 2x - \tfrac{1}{2}\cos 8x$$

$$\int \sin 5x \sin 3x \, dx = \int \left(\tfrac{1}{2}\cos 2x - \tfrac{1}{2}\cos 8x\right) dx$$

$$= \tfrac{1}{4}\sin 2x - \tfrac{1}{16}\sin 8x + C$$

Similarly: $\int \sin 7x \cos 2x \, dx$

$$\sin 7x \cos 2x + \cos 7x \sin 2x = \sin(7x+2x) = \sin 9x$$

Add:

$$\sin 7x \cos 2x - \cos 7x \sin 2x = \sin(7x-2x) = \sin 5x$$

$$2 \sin 7x \cos 2x = \sin 9x + \sin 5x$$

$$\sin 7x \cos 2x = \tfrac{1}{2}\sin 9x + \tfrac{1}{2}\sin 5x$$

$$\int \sin 7x \cos 2x \, dx = \int \left(\tfrac{1}{2}\sin 9x + \tfrac{1}{2}\sin 5x\right) dx$$

$$= -\tfrac{1}{18}\cos 9x - \tfrac{1}{10}\cos 5x + C$$

7.3 Trig substitution

ex: $\displaystyle\int \frac{x}{\sqrt{36-x^2}}\,dx$

$x = 6\sin\theta$

[NOTE: In effect, we are actually making the θ-substitution: $\theta = \sin^{-1}\frac{x}{6}$]

$dx = 6\cos\theta\, d\theta$

$36 - x^2 = 36 - 36\sin^2\theta$

$= 36(1-\sin^2\theta)$

$= 36\cos^2\theta$

$\sqrt{36-x^2} = \sqrt{36\cos^2\theta}$

$= 6\cos\theta \leftarrow 6|\cos\theta|$ but $\cos\theta$ is positive in quadrants I and IV

[This is the fine print.]

for $x = 6\sin\theta$

$\frac{x}{6} = \sin\theta$

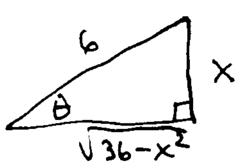
$\sin^{-1}\frac{x}{6} = \theta$ with $\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$

$$\int \frac{x}{\sqrt{36-x^2}}\,dx = \int \frac{6\sin\theta}{\cancel{6\cos\theta}}\cdot \cancel{6\cos\theta}\, d\theta$$

$$= \int 6\sin\theta\, d\theta = -6\cos\theta + C$$

$$= -6\cdot\frac{\sqrt{36-x^2}}{6} + C$$

$$= -\sqrt{36-x^2} + C$$



$\frac{x}{6} = \frac{opp}{hyp} = \sin\theta$

so $\cos\theta = \frac{adj}{hyp} = \frac{\sqrt{36-x^2}}{6}$

10) $$\int \frac{x^5}{\sqrt{x+2}}\,dx = \int \frac{x^5}{\sqrt{x^2+\sqrt{2}^2}}\,dx$$

$\tan^{-1}\frac{x}{\sqrt{2}} = \theta$ with $-\frac{\pi}{2} \leq \theta < \frac{\pi}{2}$

$x = \sqrt{2}\tan\theta$

$dx = \sqrt{2}\sec^2\theta\,d\theta$

$$\begin{aligned} x^2+\sqrt{2}^2 &= 2\tan^2\theta + 2 \\ &= 2(\tan^2\theta+1) \\ &= 2\sec^2\theta \end{aligned}$$

$$\begin{aligned} \sqrt{x^2+2} &= \sqrt{2\sec^2\theta} \\ &= \sqrt{2}\sec\theta \end{aligned}$$

$$= \int \frac{(\sqrt{2}\tan\theta)^5}{\cancel{\sqrt{2}}\,\cancel{\sec\theta}}\,\cancel{\sqrt{2}}\,\overset{\sec\theta}{\cancel{\sec^2\theta}}\,d\theta$$

$$= \int 2^{5/2}\tan^5\theta\,\sec\theta\,d\theta$$

$$= 2^{5/2}\int \tan^4\theta\,(\sec\theta\tan\theta)\,d\theta$$

$$= 2^{2+\frac{1}{2}}\int (\tan^2\theta)^2\,(\sec\theta\tan\theta)\,d\theta$$

$$= 2^2\cdot 2^{1/2}\int (\sec^2\theta - 1)^2\,(\sec\theta\tan\theta)\,d\theta$$

$$= 4\sqrt{2}\int (u^2-1)^2\,du = 4\sqrt{2}\int (u^4-2u^2+1)\,du$$

$u = \sec\theta$

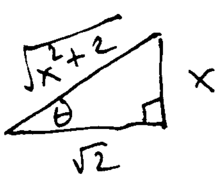
$du = \sec\theta\tan\theta\,d\theta$

$$= 4\sqrt{2}\left[\frac{1}{5}u^5 - \frac{2}{3}u^3 + u\right] + C$$

$$= 4\sqrt{2}\left[\frac{1}{5}\sec^5\theta - \frac{2}{3}\sec^3\theta + \sec\theta\right] + C$$

$$= 4\sqrt{2}\left[\frac{1}{5}\frac{(x^2+2)^{5/2}}{2^{5/2}} - \frac{2}{3}\frac{(x^2+2)^{3/2}}{2^{3/2}} + \frac{(x^2+2)^{1/2}}{2^{1/2}}\right] + C$$

$\tan\theta = \frac{x}{\sqrt{2}} = \frac{\text{opp}}{\text{adj}}$



so $\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{x^2+2}}{\sqrt{2}}$

$$= \frac{(x^2+2)^{1/2}}{2^{1/2}}$$

$$5)\quad \int_{t=\sqrt{2}}^{t=2} \frac{1}{t^3\sqrt{t^2-1}}\,dt = \int_{\theta=\frac{\pi}{4}}^{\theta=\pi/3} \frac{\sec\theta\tan\theta\,d\theta}{\sec^3\theta\tan\theta}$$

$t = \sec\theta, \quad 0 \leq \theta < \frac{\pi}{2}, \text{ or } \pi \leq \theta < \frac{3\pi}{2}$

(fine print)

$dt = \sec\theta\tan\theta\,d\theta$

$t^2 - 1 = \sec^2\theta - 1$

$= \tan^2\theta$

$\sqrt{t^2-1} = \tan\theta$

$t = \sqrt{2} \Rightarrow \sqrt{2} = \sec\theta$

$\Rightarrow \cos\theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\Rightarrow \theta = \frac{\pi}{4}$

Also $t = 2 \Rightarrow 2 = \sec\theta$

$\Rightarrow \frac{1}{2} = \cos\theta$

$\Rightarrow \theta = \cos^{-1}\frac{1}{2} = \frac{\pi}{3}$

$$= \int_{\pi/4}^{\pi/3} \frac{1}{\sec^2\theta}\,d\theta$$

$$= \int_{\pi/4}^{\pi/3} \cos^2\theta\,d\theta$$

$$= \int_{\pi/4}^{\pi/3} \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= \left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_{\theta=\pi/4}^{\pi/3}$$

$$= \frac{1}{2}\left[\frac{\pi}{3} - \frac{\pi}{4}\right] + \frac{1}{4}\left[\sin 2\left(\frac{\pi}{3}\right) - \sin 2\left(\frac{\pi}{4}\right)\right]$$

$$= \frac{1}{2}\left[\frac{4\pi}{12} - \frac{3\pi}{12}\right] + \frac{1}{4}\left[\frac{\sqrt{3}}{2} - 1\right]$$

$$= \frac{\pi}{24} + \frac{\sqrt{3}}{8} - \frac{1}{4}$$

$$= 0.0974060448$$

By calculator: (TI-84)

.0974060448

24) $\displaystyle\int \frac{dt}{\sqrt{t^2-6t+13}}$

Observe: $t^2-6t+13$ is irreducible over the reals, (that is, its zeros are non-real complex conjugates).

$$t^2-6t+9 \quad +13-9 = (t-3)^2+4 = (t-3)^2+2^2$$

$$\int \frac{dt}{\sqrt{(t-3)^2+2^2}} = \int \frac{du}{\sqrt{u^2+2^2}} = \int \frac{2\sec^2\theta\, d\theta}{2\sec\theta}$$

$u = t-3$
$du = dt$

$u = 2\tan\theta$
$du = 2\sec^2\theta\, d\theta$
$u^2+2^2 = 4\tan^2\theta + 4$
$= 4(\tan^2\theta+1)$
$= 4\sec^2\theta$
$\sqrt{u^2+2^2} = \sqrt{4\sec^2\theta}$
$= 2\sec\theta$

$$= \int \sec\theta\, d\theta$$
$$= \ln|\sec\theta + \tan\theta| + C$$
$$= \ln\left|\tfrac{1}{2}\sqrt{u^2+4} + \tfrac{1}{2}u\right| + C$$
$$= \ln\left|\tfrac{1}{2}\sqrt{(t-3)^2+4} + \tfrac{1}{2}(t-3)\right| + C$$

Using $\sec\theta = \frac{1}{2}\sqrt{u^2+4}$

$\tan\theta = \frac{u}{2}$

Remark: One can also write the answer as

$$\ln \tfrac{1}{2}\left|\sqrt{(t-3)^2+4} + (t-3)\right| + C$$
$$= \ln\left|\sqrt{(t-3)^2+4} + (t-3)\right| + \ln(1/2) + C$$
$$= \ln\left|\sqrt{(t-3)^2+4} + (t-3)\right| + C_1$$

where
$C_1 = \ln(1/2) + C$