# 7.4 Integration of Rational Functions by Partial Fraction Decomposition

GOAL: Show we can integrate any rational function

ex: Suppose a typical rational function can be made to look like

$$f(x) = x^2 + 5x + 2 + \frac{5}{x-2} + \frac{7}{(x-2)^2} + \frac{1}{(x-2)^3} + \frac{8}{x+3}$$

$$+ \frac{2x-5}{x^2+2x+5} + \frac{x-7}{(x^2+2x+5)^2}$$

Let's show we find an antiderivative of each term.

$$\int (x^2+5x+2)\,dx = \frac{1}{3}x^3 + \frac{5}{2}x^2 + 2x + C$$ Too easy.

$$\int \frac{5}{x-2}\,dx = 5\ln|x-2| + C$$

$$\int \frac{7}{(x-2)^2}\,dx = 7\int (x-2)^{-2}\,dx = -7(x-2)^{-1} + C$$
$$= \frac{-7}{x-2} + C$$

$$\int \frac{1}{(x-2)^3}\,dx = \int (x-2)^{-3}\,dx = -\frac{1}{2}(x-2)^{-2} + C = \frac{-1}{2(x-2)^2} + C$$

$$\int \frac{8}{x+3}\,dx = 8\ln|x+3| + C$$

Now, the hard parts: $\displaystyle\int \frac{2x-5}{x^2+2x+5}\,dx = ?$

Observe: $x^2+2x+5 = x^2+2x+1 + 5 - 1$
$= (x+1)^2 + 4 = (x+1)^2 + 2^2$

$$\int \frac{2x-5}{(x+1)^2+4}\,dx = \int \frac{2(u-1)-5}{u^2+4}\,du$$

$u = x+1$
$du = dx$
$u-1 = x$

$$= \int \frac{2u-7}{u^2+4}\,du$$

$$= \int \frac{2u}{u^2+4}\,du - 7\int \frac{1}{u^2+4}\,du$$

Let $w = u^2+4$
$dw = 2u\,du$

$$= \int \frac{dw}{w} - \frac{7}{2}\tan^{-1}\frac{u}{2}$$

$$= \ln|w| - \frac{7}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C$$

$$= \ln|u^2+4| - \frac{7}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C$$

$$= \ln|(x+1)^2+4| - \frac{7}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C$$

Not so easy, but we can do it!

One left: $\displaystyle\int \frac{x-7}{(x^2+2x+5)^2}\,dx = \int \frac{x-7}{[(x+1)^2+4]^2}\,dx$

$u = x+1$
$du = dx$
$u-1 = x$

$$= \int \frac{(u-1)-7}{(u^2+4)^2}\,du = \int \frac{u-8}{(u^2+4)^2}\,du$$

$$= \underbrace{\frac{1}{2}\int \frac{2u\,du}{(u^2+4)^2}}_{(1)} - \underbrace{8\int \frac{1}{(u^2+4)^2}\,du}_{(2)}$$

For (1): let $w = u^2+4$
$dw = 2u\,du$

so $\displaystyle\frac{1}{2}\int \frac{2u\,du}{(u^2+4)^2} = \frac{1}{2}\int \frac{dw}{w^2} = \frac{1}{2}\int w^{-2}\,dw$

$$= -\frac{1}{2}w^{-1} + C = -\frac{1}{2}(u^2+4)^{-1} + C = -\frac{1}{2}[(x+1)^2+4]^{-1} + C$$

Now for integral 2 : $-8\int \frac{1}{(u^2+4)^2}\,du = -8\int \frac{1}{(u^2+2^2)^2}\,du$

Let $u = 2\tan\theta$

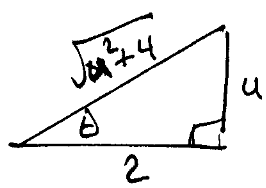
$du = 2\sec^2\theta\,d\theta$

$u^2+2^2 = 4\tan^2\theta+4$

$= 4(\tan^2\theta+1)$

$= 4\sec^2\theta$

NOTE: $\frac{u}{2} = \tan\theta = \frac{\text{opp}}{\text{adj}}$




$$= -8\int \frac{2\sec^2\theta\,d\theta}{(4\sec^2\theta)^2}$$

$$= -8\int \frac{2\sec^2\theta\,d\theta}{16\sec^4\theta}$$

$$= -\int \frac{1}{\sec^2\theta}\,d\theta = -\int \cos^2\theta\,d\theta$$

$$= -\int \left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)d\theta$$

$$= -\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right] + C$$

$$= -\left[\frac{1}{2}\theta + \frac{1}{4}\cdot 2\sin\theta\cos\theta\right] + C$$

$$= -\frac{1}{2}\theta - \frac{1}{2}\sin\theta\cos\theta + C = -\frac{1}{2}\tan^{-1}\frac{u}{2} - \frac{1}{2}\,\frac{u}{\sqrt{u^2+4}}\cdot\frac{2}{\sqrt{u^2+4}} + C$$

$$= -\frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) - \frac{u}{u^2+4} + C$$

$$= -\frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) - \frac{x+1}{(x+1)^2+4} + C$$

Remark: (1) Put this all together and we've integrated the original rational function.

(2) BUT we need to show how to find the partial fraction decomposition of a rational function.

<u>Form</u> of the P.F.D.

<u>Remark</u>: We can always do long division to make an improper rational function into the sum of a polynomial and a <u>proper</u> rational function.

<u>ex</u>: $$\frac{x^3+5x^2+2x+1}{x^2+4} = x+5 + \frac{-2x-19}{x^2+4}$$

$$\begin{array}{r l}
 & \quad x + 5 \\
x^2+4 \,\big) & \overline{x^3+5x^2+2x+1} \\
 & \underline{x^3 \qquad\quad +4x} \\
 & \qquad 5x^2-2x+1 \\
 & \qquad \underline{5x^2 \qquad +20} \\
 & \qquad\qquad -2x-19
\end{array}$$

3a) $$\frac{x^4+1}{x^5+4x^3} = \frac{x^4+1}{x^3(x^2+4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{x^2+4}$$

5b) $$\frac{x^4}{(x^2-x+1)(x^2+2)^2} = \frac{Ax+B}{x^2-x+1} + \frac{Cx+D}{x^2+2} + \frac{Ex+F}{(x^2+2)^2}$$

10) $\displaystyle\int \frac{y}{(y+4)(2y-1)}\,dy$

$$\frac{y}{(y+4)(2y-1)} = \frac{A}{y+4} + \frac{B}{2y-1}$$ Multiply by the LCD:

$$y = \frac{A(y+4)(2y-1)}{y+4} + \frac{B(y+4)(2y-1)}{2y-1}$$

$$y = A(2y-1) + B(y+4)$$

Idea: If these two polynomial functions are the same, they should evaluate the same way.

let $y=-4$: $\quad -4 = A[2(-4)-1] + B\underbrace{(-4+4)}_{0}$

$-4 = A(-9) \Rightarrow \boxed{A = \frac{4}{9}}$

let $y=\frac{1}{2}$: $\quad \frac{1}{2} = A\underbrace{(2\cdot\tfrac{1}{2}-1)}_{0} + B(\frac{1}{2}+4)$

$\frac{1}{2} = \frac{9}{2}B \Rightarrow \boxed{B = \frac{1}{9}}$

so $\displaystyle\frac{y}{(y+4)(2y-1)} = \frac{4}{9}\cdot\frac{1}{y+4} + \frac{1}{9}\cdot\frac{1}{2y-1}$

and $\displaystyle\int \frac{y\,dy}{(y+4)(2y-1)} = \frac{4}{9}\int\frac{dy}{y+4} + \frac{1}{9}\int\frac{dy}{2y-1}$

$\displaystyle = \frac{4}{9}\ln|y+4| + \frac{1}{9}\cdot\frac{1}{2}\ln|2y-1| + C$

$\displaystyle = \frac{4}{9}\ln|y+4| + \frac{1}{18}\ln|2y-1| + C$