§7.4 38) $\displaystyle\int \frac{x^3+2x^2+3x-2}{(x^2+2x+2)^2}\,dx$

$$\text{integrand} = \frac{Ax+B}{x^2+2x+2} + \frac{Cx+D}{(x^2+2x+2)^2}$$

To both sides of the equation, Multiply by LCD $= (x^2+2x+2)^2$. This will clear the denominators

$$x^3+2x^2+3x-2 = (Ax+B)(x^2+2x+2) + Cx+D$$

scratch work:

$$\begin{array}{r} x^2+2x+2 \\ Ax+B \\ \hline Bx^2+2Bx+2B \\ Ax^3+2Ax^2+2Ax \\ \hline Ax^3+(2A+B)x^2+(2A+2B)x+2B \end{array}$$

$$x^3+2x^2+3x-2 = Ax^3+(2A+B)x^2+(2A+2B+C)x+(2B+D)$$

In order for these two cubic polynomials to be equal,

Principle: the x^3 coefficients on both sides must be equal,

" x^2 " " " " " " "

" x " " " " " " "

" constant " " " " " " "

$$\begin{cases} 1 = A \\ 2 = 2A + B \\ 3 = 2A+2B+C \\ -2 = 2B + D \end{cases}$$

This system of linear equations can be represented by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 2 \\ 2 & 2 & 1 & 0 & 3 \\ 0 & 2 & 0 & 1 & -2 \end{bmatrix}$$

[By TI84] This matrix is row-equivalent to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

so $A=1$
$B=0$
$C=1$
$D=-2$

38 cont'd) $\int \left[\frac{x}{x^2+2x+2} + \frac{x-2}{(x^2+2x+2)^2} \right] dx$ and we know how to do this!* So there!

[* To be completed later in the notes.]

4.4 L'Hospital's Rule (briefly)

Main idea: If $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ OR $\frac{\infty}{\infty}$

then $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$.

ex: $\lim\limits_{x \to 2} \frac{x^2-4}{x^2-3x+2}$ seems to equal $\frac{2^2-4}{2^2-3\cdot 2+2} = \frac{0}{0}$

$$\lim_{x \to 2} \frac{x^2-4}{x^2-3x+2} = \lim_{x \to 2} \frac{\frac{d}{dx}[x^2-4]}{\frac{d}{dx}[x^2-3x+2]}$$

$$= \lim_{x \to 2} \frac{2x}{2x-3} = \frac{2\cdot 2}{2\cdot 2-3} = \frac{4}{1} = 4$$

Alternatively: $\frac{x^2-4}{x^2-3x+2} = \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}(x-1)}$

$$\lim_{x \to 2} \frac{x+2}{x-1} = \frac{4}{1} = 4$$

ex: $\lim\limits_{x \to 0} \frac{x^2}{1-\cos x} = \lim\limits_{x \to 0} \frac{2x}{\sin x} = \lim\limits_{x \to 0} \frac{2}{\cos x} = \frac{2}{1} = 2$

7.8 Improper integrals

ex: A car is moving but slowing down.
Its speed is $\frac{1}{t^2}$ meters/second.
So at $t = 1$ second, its speed is 1 m/s
but at $t = 10$ s, its speed is $\frac{1}{100}$ m/s
How far does the car move between $t=1$ and $t=10$?

$$\text{distance} = \int_{t=1}^{t=10} \left(\frac{ds}{dt}\right) dt \qquad \text{where } \frac{ds}{dt} = \text{speed} = \frac{1}{t^2}$$

$$= \int_1^{10} \frac{1}{t^2}\,dt = \int_1^{10} t^{-2}\,dt = -t^{-1}\Big]_1^{10}$$

$$= -\frac{1}{10} + \frac{1}{1} = 1 - \frac{1}{10} = \frac{9}{10} \text{ meter.}$$

ex: Between $t=1$ and $t = 1000$ seconds?

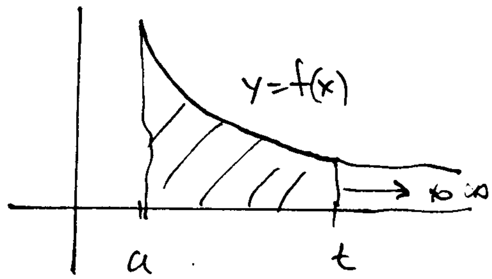
$$\int_1^{1000} \frac{1}{t^2}\,dt = 1 - \frac{1}{1000} = \frac{999}{1000} \text{ meters}$$

ex: Between $t=1$ and $t=\infty$?

$$\int_1^{\infty} \frac{1}{t^2}\,dt = 1 - \frac{1}{\infty} = 1 - 0 = 1 \text{ meter.}$$

[Abuse of notation]

Defn: $\displaystyle\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$



$\displaystyle\int_a^t f(x)\,dx$ = area under curve between $x=a$ and $x=t$

If $\displaystyle\lim_{t\to\infty}\int_a^t f(x)\,dx = L$

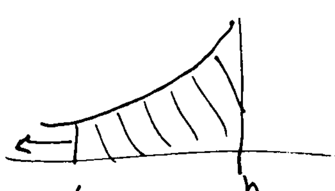
we say the improper integral <u>converges</u> <u>to</u> L.

<u>ex</u>: $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx$

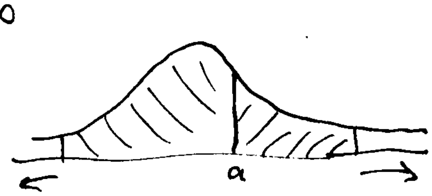
$\displaystyle = \lim_{t\to\infty}\left[-\frac{1}{x}\right]_1^t = \lim_{t\to\infty}\left[\frac{-1}{t}+\frac{1}{1}\right]$

$\displaystyle = \lim_{t\to\infty}\left[1-\frac{1}{t}\right] = 1-0 = 1$

<u>Defn</u>: $\displaystyle\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$



<u>Defn</u>. $\displaystyle\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$ provided <u>both</u> integrals on the right side converge.



ex: Now, our car slows but according to the function $\frac{ds}{dt} = \frac{1}{t}$ meters/second. How far does the car roll between $t = 1$ sec. and $t =$ forever?

$$\int_{t=1}^{\infty} \left(\frac{ds}{dt}\right) dt = \int_{1}^{\infty} \frac{1}{t}\, dt = \lim_{b \to \infty} \int_{1}^{b} t^{-1} dt$$

$$= \lim_{b \to \infty} \Big[\ln|t|\Big]_{1}^{b} = \lim_{b \to \infty} \Big[\ln b - \ln 1\Big]$$

$$= \lim_{b \to \infty} \ln b = \infty$$ "The integral is divergent."

ex: You win the lottery. Your award is 1 dollar/year. But the rate of inflation is 5% so that the present value of 1 dollar in t years is

$$f(t) = 1 \cdot e^{-.05t}$$

What is the present value of this income stream?

$$\text{present value} = \int_{0}^{\infty} f(t)\, dt = \int_{0}^{\infty} e^{-.05t}\, dt$$

$$\int_{0}^{b} e^{-.05t}\, dt = \frac{1}{-.05} e^{-.05t}\Big]_{0}^{b} = -20 e^{-.05b} + 20 e^{0}$$

$$= 20\left(1 - e^{-.05b}\right) \longrightarrow 20(1-0)$$

$= 20$ dollars.

as $b \to \infty$

NOTES ADDED AFTER CLASS

... completion of §7.4 #38)

$$\int \frac{x\,dx}{x^2+2x+2} + \int \frac{x-2}{(x^2+2x+2)^2}\,dx$$

(1) (2)

Note that $x^2+2x+2 = (x^2+2x+1)+1 = (x+1)^2+1$

To calculate integral (1):

$$\int \frac{x\,dx}{(x+1)^2+1} = \int \frac{u-1}{u^2+1}\,du = \int \frac{u\,du}{u^2+1} - \int \frac{du}{u^2+1}$$

let $u = x+1$
$u-1 = x$
$du = dx$

"in-line" substitution

$$= \frac{1}{2}\int \frac{2u\,du}{u^2+1} - \tan^{-1} u = \frac{1}{2}\int \frac{d(u^2+1)}{u^2+1} - \tan^{-1} u$$

$$= \frac{1}{2}\ln|u^2+1| - \tan^{-1} u + C = \frac{1}{2}\ln|x^2+2x+2| - \tan^{-1}(x+1) + C$$

To calculate integral (2):

$$\int \frac{x-2}{[(x+1)^2+1]^2}\,dx = \int \frac{(u-1)-2}{(u^2+1)^2}\,du = \int \frac{u-3}{(u^2+1)^2}\,du$$

$$= \frac{1}{2}\int \frac{2u\,du}{(u^2+1)^2} - 3\int \frac{du}{(u^2+1)^2}$$

(2a) (2b)

For integral (2a):

$$\frac{1}{2}\int \frac{2u\,du}{(u^2+1)^2} = \frac{1}{2}\int \frac{dw}{w^2} = \frac{1}{2}\int w^{-2}\,dw = -\frac{1}{2}w^{-1} + C = \frac{-1}{2w} + C$$

let $w = u^2+1$
$dw = 2u\,du$

$$= \frac{-1}{2(u^2+1)} = \frac{-1}{2[(x+1)^2+1]} = \frac{-1}{2(x^2+2x+2)}$$

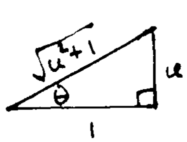
For integral (2b):

$$-3\int \frac{du}{(u^2+1)^2} = -3\int \frac{\sec^2\theta\,d\theta}{(\sec^2\theta)^2} = -3\int \frac{d\theta}{\sec^2\theta} = -3\int \cos^2\theta\,d\theta$$

let $u = \tan\theta$
$du = \sec^2\theta\,d\theta$
$u^2+1 = \tan^2\theta + 1 = \sec^2\theta$

$$= -3\int \frac{1+\cos 2\theta}{2}\,d\theta = -\frac{3}{2}\int d\theta - \frac{3}{2}\int \cos 2\theta\,d\theta$$

$$= -\frac{3}{2}\theta - \frac{3}{4}\sin 2\theta + C = -\frac{3}{2}\theta - \frac{3}{4}\cdot 2\sin\theta\cos\theta + C$$



$$= -\frac{3}{2}\tan^{-1} u - \frac{3}{2}\,\frac{u}{\sqrt{u^2+1}}\cdot\frac{1}{\sqrt{u^2+1}} + C = -\frac{3}{2}\tan^{-1} u - \frac{3}{2}\cdot\frac{u}{u^2+1} + C$$

$\tan^{-1} u = \theta$
$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$= -\frac{3}{2}\tan^{-1}(x+1) - \frac{3}{2}\,\frac{(x+1)}{(x+1)^2+1} + C$$

$$= -\frac{3}{2}\tan^{-1}(x+1) - \frac{(3x+3)}{2(x^2+2x+2)} + C$$

continued ...

§7.4 #38 continued)

$$\{\text{integral } ②\} = \int \frac{x-2}{(x^2+2x+2)^2}\,dx = \{\text{integral } ②a\} + \{\text{integral } ②b\}$$

$$= \frac{-1}{2(x^2+2x+2)} - \frac{3}{2}\tan^{-1}(x+1) - \frac{(3x+3)}{2(x^2+2x+2)} + C$$

$$= -\frac{3}{2}\tan^{-1}(x+1) - \frac{3x+4}{2(x^2+2x+2)} + C$$

Answer $= \{\text{integral } ①\} + \{\text{integral } ②\}$

$$= \frac{1}{2}\ln|x^2+2x+2| - \tan^{-1}(x+1) - \frac{3}{2}\tan^{-1}(x+1) - \frac{3x+4}{2(x^2+2x+2)} + C$$

$$= \frac{1}{2}\ln|x^2+2x+2| - \frac{5}{2}\tan^{-1}(x+1) - \frac{3x+4}{2(x^2+2x+2)} + C$$