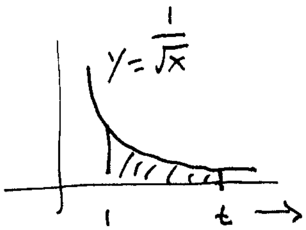
7.8 Improper integrals (continued)

ex: $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ Does the integral converge? If so, to what value?

$$= \lim_{t\to\infty} \int_1^t \frac{1}{x^{1/2}}\,dx$$



$$\int_1^t x^{-1/2}\,dx = 2x^{1/2}\Big]_1^t = 2\cdot t^{1/2} - 2\cdot 1^{1/2}$$
$$= 2\sqrt{t} - 2$$

$$\lim_{t\to\infty}\int_1^t x^{-1/2}\,dx = \lim_{t\to\infty}\left[2\sqrt{t}-2\right] = \infty$$

The integral is $\boxed{\text{DIVERGENT}}$.

ex: $\int_4^\infty \frac{1}{x^{3/2}}\,dx = ?$ $\quad\int_4^t x^{-3/2}\,dx = -2x^{-1/2}\Big]_4^t$

$$= \frac{-2}{\sqrt{t}} + \frac{2}{\sqrt{4}} = \frac{-2}{\sqrt{t}} + 1$$

$$\int_4^\infty \frac{1}{x^{3/2}}\,dx = \lim_{t\to\infty}\left(\frac{-2}{\sqrt{t}} + 1\right) = 0 + 1 = 1$$

more generally

$\int_1^\infty \frac{1}{x^p}\,dx$ is convergent if $p>1$ and divergent if $p\le 1$.

ex: $\int_1^\infty \frac{1}{x^{1.01}}\,dx$ converges. But $\int_1^\infty \frac{1}{x^{0.99}}\,dx$ diverges.

How to abuse notation in improper integrals

[Don't tell anyone.] Pretend that ∞ is just a big number.

ex: $$\int_1^\infty \frac{1}{x^{1.01}}\,dx = \int_1^\infty x^{-1.01}\,dx$$

$$= \frac{1}{-0.01}\, x^{-0.01}\Big]_1^\infty$$

$$= -100 \left.\frac{1}{\sqrt[100]{x}}\right|_1^\infty = -100\,\frac{1}{\sqrt[100]{\infty}} + \frac{100}{\sqrt[100]{1}}$$

$$= -0 + 100 = 100$$

13) $$\int_{-\infty}^{\infty} x e^{-x^2}\,dx \overset{①}{=} \int_0^\infty x e^{-x^2}\,dx \overset{②}{+} \int_{-\infty}^{0} x e^{-x^2}\,dx$$

① $$\int_{x=0}^{t} x e^{-x^2}\,dx = -\frac{1}{2}\int_0^t -2x e^{-x^2}\,dx$$

$$\boxed{\begin{aligned} u &= -x^2 \\ du &= -2x\,dx \\ x=0 &\Leftrightarrow u=0 \\ x=t &\Leftrightarrow u=-t^2 \end{aligned}}$$

$$= -\frac{1}{2}\int_{u=0}^{u=-t^2} e^u\,du$$

$$= -\frac{1}{2}\left[e^{-t^2} - e^0\right] = -\frac{1}{2}\left(e^{-t^2}-1\right)$$

$$= \frac{1}{2}\left(1-e^{-t^2}\right) = \frac{1}{2}\left(1-\frac{1}{e^{t^2}}\right)$$

$$\int_0^\infty x e^{-x^2}\,dx = \lim_{t\to\infty} \frac{1}{2}\left(1-\frac{1}{e^{t^2}}\right) = \frac{1}{2}$$

(2) $$\int_{-\infty}^{0} x e^{-x^2}\,dx = ?$$

$u=-x^2$

$du=-2x\,dx$

$x=0 \Leftrightarrow u=0$

$x=t \Leftrightarrow u=-t^2$

$$\int_t^0 x e^{-x^2}\,dx = \frac{-1}{2}\int_t^0 -2x e^{-x^2}\,dx$$

$$= -\frac{1}{2}\int_{-t^2}^{0} e^u\,du = -\frac{1}{2}\left[e^0 - e^{-t^2}\right]$$

$$= \frac{1}{2}\left(\frac{1}{e^{t^2}} - 1\right)$$

$$\lim_{t\to-\infty}\int_t^0 x e^{-x^2}\,dx = \lim_{t\to-\infty}\frac{1}{2}\left(\frac{1}{e^{t^2}}-1\right) = -\frac{1}{2}$$

$$\text{Answer} = \int_0^{\infty} x e^{-x^2}\,dx + \int_{-\infty}^{0} x e^{-x^2}\,dx = \frac{1}{2} + -\frac{1}{2} = 0$$

ex: $$\int_0^{\infty} x^2 e^{-x}\,dx = \lim_{t\to\infty}\int_0^t x^2 e^{-x}\,dx$$

$$\int_0^t x^2 e^{-x}\,dx = -x^2e^{-x}\Big|_0^t - 2xe^{-x}\Big|_0^t - 2e^{-x}\Big|_0^t$$

sgn	diff	int
$+$	x^2	e^{-x}
$-$	$2x$	$-e^{-x}$
$+$	2	e^{-x}
	0	$-e^{-x}$

$$= \left(-t^2e^{-t} + 0\right) + \left(-2te^{-t} + 0\right) + \left(-2e^{-t} + 2e^0\right)$$

$$\lim_{t\to\infty} \int_0^t x^2 e^{-x}\,dx$$

$$= \lim_{t\to\infty} \left(-t^2 e^{-t} - 2t e^{-t} - 2e^{-t} + 2 \right)$$

$$= \lim_{t\to\infty} (-t^2 - 2t - 2)\, e^{-t} + 2$$

$$= \lim_{t\to\infty} \frac{-t^2 - 2t - 2}{e^t} + 2$$

$$\underset{\text{L'Hospital's Rule}}{=} \lim_{t\to\infty} \frac{-2t - 2}{e^t} + 2$$

$$\underset{\text{L'H}}{=} \lim_{t\to\infty} \frac{-2}{e^t} + 2 = 0 + 2 = \boxed{2}$$

Added after class: Another example of an integral of a rational function making use of its partial fraction decomposition:

ex: Evaluate $\displaystyle\int \frac{3x^2+10x+4}{(x+4)^2(x-2)}\,dx$.

$$\frac{3x^2+10x+4}{(x+4)^2(x-2)} = \frac{A}{x+4} + \frac{B}{(x+4)^2} + \frac{C}{x-2}$$ where A, B, and C are constants to be determined.

Multiply both sides of the equation by the LCD $= (x+4)^2(x-2)$:

$$3x^2+10x+4 = \frac{A(x+4)^2(x-2)}{x+4} + \frac{B(x+4)^2(x-2)}{(x+4)^2} + \frac{C(x+4)^2(x-2)}{x-2}$$

$$3x^2+10x+4 = A(x+4)(x-2) + B(x-2) + C(x+4)^2$$

If these two quadratic functions are to be equal, they will take on the same values for each possible value of x. Select three (strategically chosen) values of x.

Let $x=-4$: $\quad 3(-4)^2+10(-4)+4 = A(-4+4)(-4-2) + B(-4-2) + C(-4+4)^2$ or

$$12 = -6B \Rightarrow \boxed{B=-2}$$

Let $x=2$: $\quad 3\cdot 2^2+10\cdot 2+4 = A(2+4)(2-2) + B(2-2) + C(2+4)^2$ or

$$36 = 36C \Rightarrow \boxed{C=1}$$

Let $x=0$: $\quad 3\cdot 0^2+10\cdot 0+4 = A(0+4)(0-2) + B(0-2) + C(0+4)^2$ or

$$4 = -8A - 2B + 16C = -8A - 2(-2) + 16(1) = -8A + 20$$

$$\Rightarrow -16 = -8A \Rightarrow \boxed{A=2}$$

$$\int \frac{3x^2+10x+4}{(x+4)^2(x-2)}\,dx = \int \frac{2}{x+4}\,dx + \int \frac{-2}{(x+4)^2}\,dx + \int \frac{1}{x-2}\,dx$$

$$= 2\int (x+4)^{-1}\,dx - 2\int (x+4)^{-2}\,dx + \int (x-2)^{-1}\,dx$$

$$= 2\ln|x+4| + 2(x+4)^{-1} + \ln|x-2| + C$$