A couple of loose ends

end of § 7.4 Rationalizing substitutions

39) $$\int \frac{\sqrt{x+1}}{x}\,dx = \int \frac{u \cdot 2u\,du}{u^2-1} = \int \frac{2u^2\,du}{u^2-1}$$

Try $u = \sqrt{x+1}$

$u^2 = x+1$

$u^2 - 1 = x$

$2u\,du = dx$

$$\frac{2u^2}{u^2-1} = \frac{2u^2-2+2}{u^2-1} = \frac{2u^2-2}{u^2-1} + \frac{2}{u^2-1}$$

$$= \frac{2\cancel{(u^2-1)}}{\cancel{u^2-1}} + \frac{2}{u^2-1} = 2 + \frac{2}{u^2-1}$$

[This is, effectively, long division.]

$$\frac{2}{u^2-1} = \frac{2}{(u-1)(u+1)} = \frac{A}{u-1} + \frac{B}{u+1}$$

$$2 = A(u+1) + B(u-1)$$

Let $u=1$: $2 = 2A \Rightarrow A = 1$

$u=-1$: $2 = -2B \Rightarrow B = -1$

so $\dfrac{2}{u^2-1} = \dfrac{1}{u-1} - \dfrac{1}{u+1}$

$$\int \frac{2u^2\,du}{u^2-1} = \int \left(2 + \frac{1}{u-1} - \frac{1}{u+1}\right) du = 2u + \ln|u-1| - \ln|u+1| + C$$

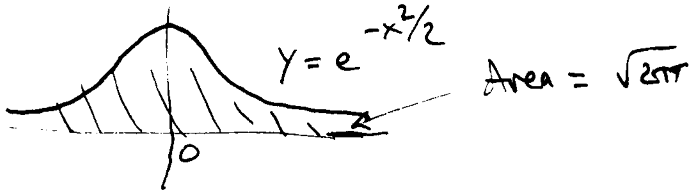
$$= 2\sqrt{x+1} + \ln\left|\sqrt{x+1} - 1\right| - \ln\left|\sqrt{x+1} + 1\right| + C$$

Next loose end:

§ 7.8 Improper integrals of Type 2

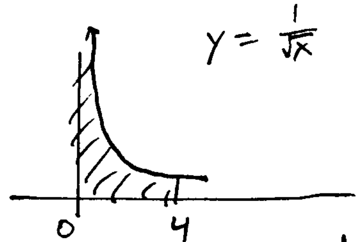
we have seen "Type 1" improper integrals: $x = \infty$ or $-\infty$.

example: $\displaystyle\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}$ [see the proof of this in Calc III]



"Type 2" is where $y = \infty$ or $-\infty$.

ex: $\displaystyle\int_0^4 \frac{1}{\sqrt{x}}\,dx$



has an infinite discontinuity at $x = 0$. So we cannot use the Fundamental Thm of Calculus.

Defn: i) If f is continuous on $(a, b]$ with discontinuity at a, we define $\displaystyle\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$ if the limit exists

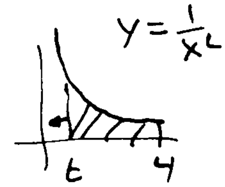
example (continued) $\displaystyle\int_t^4 x^{-1/2}\,dx = 2x^{1/2}\Big]_t^4$

$= 2 \cdot 4^{1/2} - 2 \cdot t^{1/2}$

$= 4 - 2\sqrt{t} \to 4 - 0 = \boxed{4}$ as $t \to 0^+$

ex: [the improper integral doesn't converge]

$$\int_0^4 \frac{1}{x^2}\,dx = ?$$



$$\int_t^4 x^{-2}\,dx = -x^{-1}\Big]_t^4 = -4^{-1}+t^{-1} = -\frac{1}{4}+\frac{1}{t}$$

$$\lim_{t\to 0^+}\int_t^4 \frac{1}{x^2}\,dx = \lim_{t\to 0^+} \frac{1}{t}-\frac{1}{4} = \infty$$ The integral diverges.

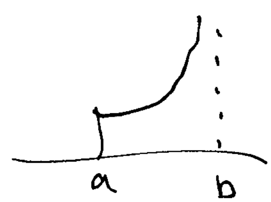
Fact: ① $\int_0^1 \frac{1}{x^p}\,dx$ diverges if $p \geq 1$ e.g. $\frac{1}{x^2}$

converges if $p<1$ e.g. $\frac{1}{\sqrt{x}}$

compare with ② $\int_1^\infty \frac{1}{x^p}\,dx$ converges if $p>1$ e.g. $\frac{1}{x^2}$

diverges if $p \leq 1$ e.g. $\frac{1}{x}$

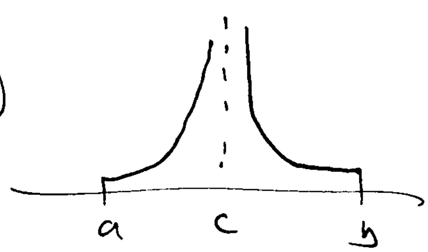
Defn ii)



$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$$

if f is discontinuous at $x=b$.

iii)



$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

by (ii) by (i)

If f has a discontinuity at $x=c$, where $a<c<b$

To say this integral converges, both integrals on the right side must converge.

Intro to differential equations [Read sections 9.1 and 9.2]

ex. [Quadratic equation] $x^2 - 8x + 15 = 0$

what are some solutions to this? $x = 3$ and $x = 5$.

check: (or verify) $3^2 - 8(3) + 15 = 9 - 24 + 15 = 0$ ✓

ex: [Differential equation] $\frac{d^2y}{dx^2} + 4y = 0$

verify that $y = \sin 2x$ satisfies this differential equation.

Remark: It is understood that y is standing for a function (of x) instead of a number.

verification: $y = \sin 2x$

$$y' = \frac{dy}{dx} = 2\cos 2x$$

$$y'' = \frac{d^2y}{dx^2} = -4\sin 2x$$

$$\frac{d^2y}{dx^2} + 4y \overset{?}{=} 0$$

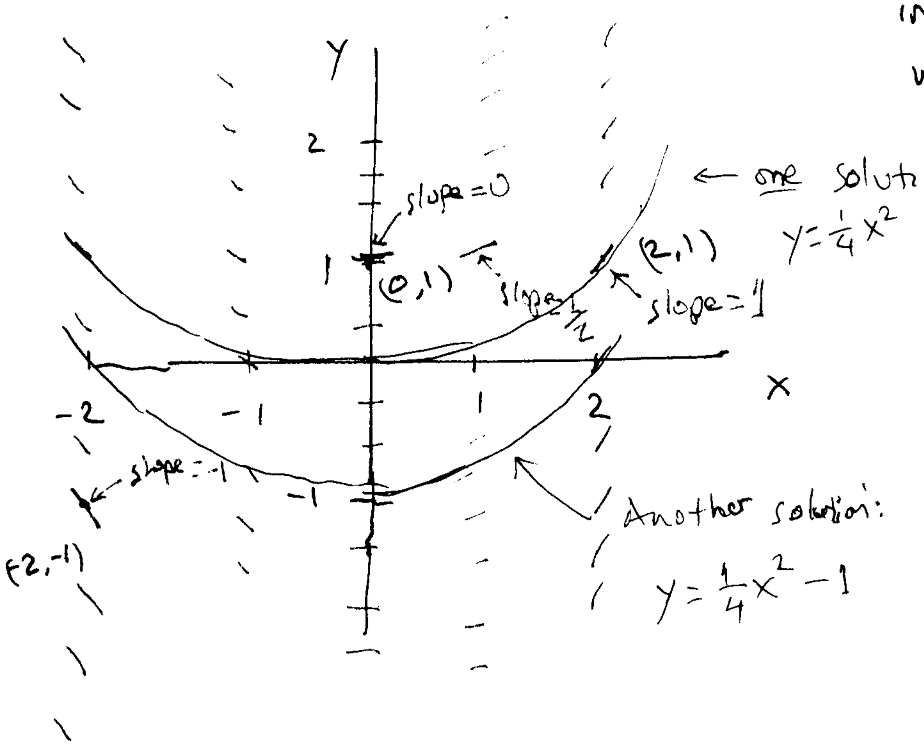
$$(-4\sin 2x) + 4(\sin 2x) \overset{\checkmark}{=} 0$$

The constant function, $y = 0$.

addition of functions

Remark: This is called a second order differential equation because it involves a second derivative.

ex: $\frac{dy}{dx} = \frac{1}{2}x$ ← By writing this, we are imagining that there is some function $y = f(x)$ with the property that its derivative (slope of tangent line; instantaneous rate of change; or whatever) is $\frac{1}{2}x$.



when $(x,y) = (2,1)$

$\frac{dy}{dx}$ should equal $\frac{1}{2}(2) = \underline{1}$

At $(x,y) = (-2,-1)$

$$\frac{dy}{dx} = \frac{1}{2}(-2) = -1$$

At $(x,y) = (0,1)$

$$\frac{dy}{dx} = \frac{1}{2}(0) = 0$$

We know the antiderivative of $\frac{1}{2}x$ is $\frac{1}{4}x^2 + C$:

$$y = \int \frac{dy}{dx}\,dx = \int \frac{1}{2}x\,dx = \frac{1}{4}x^2 + C$$

ex: Suppose you have more information, namely, when $x = 2$, $y = 0$. What does that tell us about C?

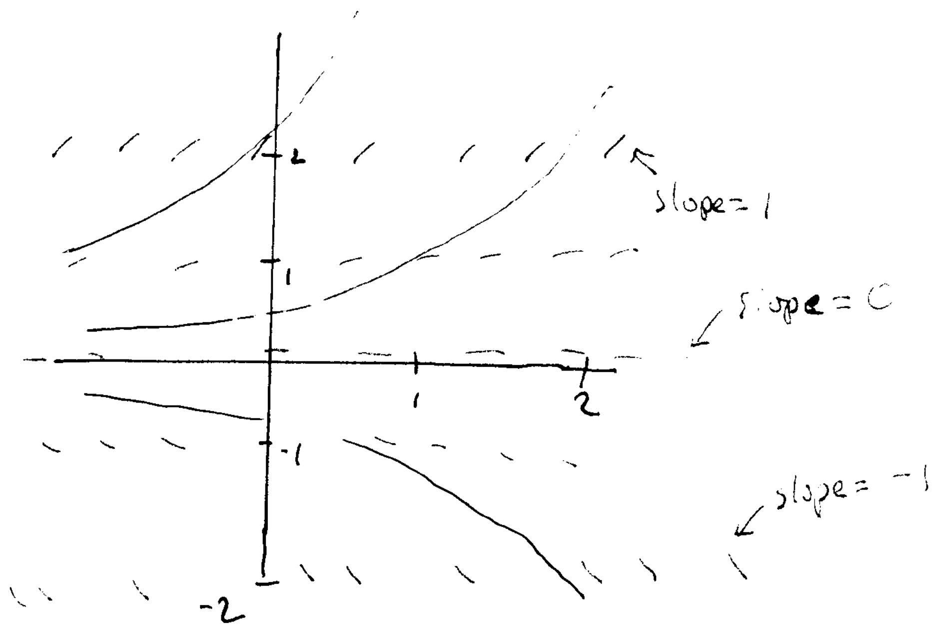
"Initial condition"

$$y = \frac{1}{4}x^2 + C$$
$$0 = \frac{1}{4}\cdot 2^2 + C \Rightarrow 0 = 1 + C \Rightarrow C = -1 \quad \text{so}$$

$y = \frac{1}{4}x^2 - 1$ is a solution of the differential equation AND the initial condition

9.3 Separable Differential Equations

ex. $\frac{dy}{dx} = \frac{1}{2}y$



$\frac{dy}{dx} = \frac{1}{2}y$ multiply both sides by $2dx$ and Divide both sides by y.

$\frac{2}{y}dy = dx$

$\int \frac{2}{y}dy = \int dx$

$2\ln|y| + C_2 = x + C_1$

$2\ln|y| = x + \underbrace{(C_1 - C_2)}_{C_3}$

$2\ln|y| = x + C_3$

$\ln|y| = \frac{1}{2}x + \frac{1}{2}C_3$

$|y| = e^{\frac{1}{2}x + \frac{1}{2}C_3}$

$|y| = e^{\frac{1}{2}C_3} \cdot e^{x/2}$

$y = \pm e^{C_3/2} \cdot e^{x/2}$

$\boxed{y = Ce^{x/2}}$

Remark: If, additionally, you had some initial condition like, say, $y(0) = 2$, then you could use that to determine that $C = 2$ and $\boxed{y = 2e^{x/2}}$ would be the solution to the "initial value problem."