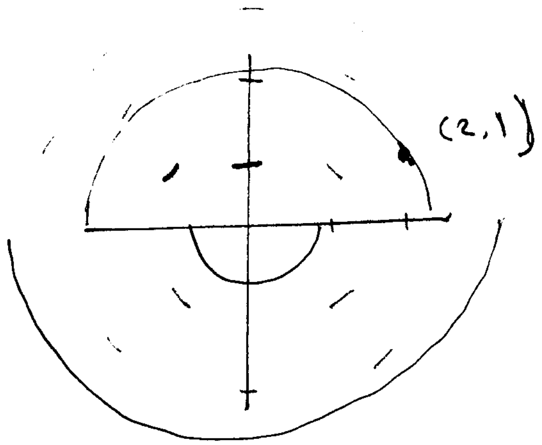
9.3 Separable Differential Equations

ex: $\dfrac{dy}{dx} = -\dfrac{x}{y}$

(x,y)	$\frac{dy}{dx}$
$(2,1)$	$-\frac{2}{1} = -2$
$(0,1)$	$-\frac{0}{1} = 0$
$(-1,1)$	$-\frac{(-1)}{1} = 1$



$$y\,dy = -x\,dx$$

$$\int y\,dy = \int -x\,dx$$

$$\tfrac{1}{2}y^2 = -\tfrac{1}{2}x^2 + C$$

$$y^2 = -x^2 + 2C \qquad \text{Let } 2C = C_1$$

$$y^2 = C_1 - x^2$$

$$y = \pm\sqrt{C_1 - x^2}$$

ex: Suppose, in addition, you have the "initial condition" $y(2)=1$.
That is, when $x=2$ we want $y=1$.

$$1 = \pm\sqrt{C_1 - 2^2}$$

$$1 = C_1 - 4$$

$$5 = C_1$$

$$\boxed{y = \sqrt{5 - x^2}}$$

is the solution of the initial value problem: $\dfrac{dy}{dx} = -\dfrac{x}{y}$, $y(2) = 1$.

Check: If $x = 2$, $y = \sqrt{5-2^2} = \sqrt{1} = 1$ ✓

Check: $\frac{dy}{dx} \stackrel{?}{=} -\frac{x}{y}$

$$y = \sqrt{5-x^2} = (5-x^2)^{1/2}$$

$$\frac{dy}{dx} = \frac{1}{2}(5-x^2)^{-1/2} \cdot \frac{d}{dx}[5-x^2]$$

$$= \frac{1}{2}(5-x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{5-x^2}}$$

On the other hand, Equal,

$$-\frac{x}{y} = -\frac{x}{\sqrt{5-x^2}}$$

so $y = \sqrt{5-x^2}$ satisfies the differential equation.

12) Solve the IVP (initial value problem)

$$\frac{dy}{dx} = \frac{\ln x}{xy}, \quad y(1) = 2$$

$$\int y\,dy = \int \frac{\ln x}{x}\,dx$$ ← OR let $u = \ln x$, $du = \frac{1}{x}dx$

$$\frac{1}{2}y^2 = \int \ln x \, d(\ln x)$$

$$\frac{1}{2}y^2 = \frac{1}{2}(\ln x)^2 + C$$

Now, when $x=1$, $y=2$:

$$\frac{1}{2}\cdot 2^2 = \frac{1}{2}(\ln 1)^2 + C$$

$$2 = C$$

$$\frac{1}{2}y^2 = \frac{1}{2}(\ln x)^2 + 2$$

$$y^2 = (\ln x)^2 + 4$$

$$y = \sqrt{(\ln x)^2 + 4}$$

9.4 Population (and money) growth

$$\frac{dP}{dt} = kP \quad , \quad P(0) = P_0$$

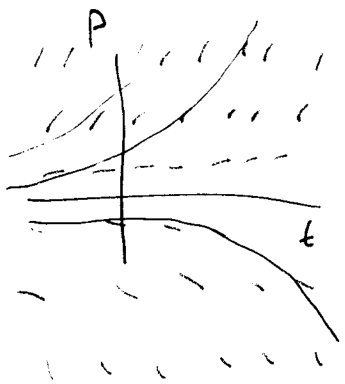
where k and P_0 are constants,

$P(t)$ = population (or bank balance)

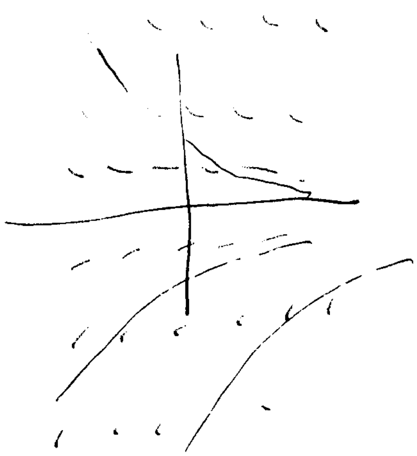
t = time

Note: $k > 0$ means growth

$k < 0$ means decay



$K > 0$



$k < 0$

Derivation of solution:

$$\frac{dP}{dt} = kP \quad , \quad P(0) = P_0$$

$$\int \frac{dP}{P} = \int k\,dt$$

$$\ln|P| = kt + C_1$$

$$|P| = e^{kt + C_1} = e^{C_1} e^{kt}$$

$$P(t) = \pm e^{C_1} e^{kt} = C e^{kt}$$

$$P_0 = P(0) = C e^0 = C \quad \Rightarrow C = P_0$$

$$\boxed{P(t) = P_0 e^{kt}}$$

Remark. So now look at some of your precalculus exponential growth (or decay) problems in a new light.

ex: You invest in a fund which pays 10% interest annually, compounded continuously. At time $t=0$, your balance $y(0)=0$. But you're able to invest \$2000 per year (continuously). How much do you have after 20 years?

How do we model this?

t = time (years)
$y(t)$ = balance (dollars)

Scenario #1: Deposit \$2000/year into a pillow case.

$$\frac{dy}{dt} = 2000 \quad , \quad y(0)=0$$

$$\int dy = \int 2000\,dt$$

$$y = 2000t + C \qquad \text{But } 0 = 2000(0)+C \Rightarrow C=0$$

$$y(t) = 2000t$$

Scenario #2: Initially, you have \$10,000 but you make no further deposits.

$$\frac{dy}{dt} = 0.10y \quad , \quad y(0) = 10{,}000$$

[This was covered by a previous example]

$$y(t) = 10{,}000\,e^{.10t}$$

Scenario #3: Instead of contributing \$2000/year into a pillow case, you are contributing into the fund from scenario #2.

$$\frac{dy}{dt} = 0.10y + 2000 \quad , \quad y(0) = 0$$

$$\frac{dy}{dt} = 0.10(y + 20{,}000)$$

$$\int \frac{dy}{y+20{,}000} = \int 0.10\,dt$$

$$\ln|y+20{,}000| = 0.10t + C$$

$$|y+20{,}000| = e^{.10t+C} = e^{C}\cdot e^{.10t}$$

$$y + 20{,}000 = \pm e^{C} e^{.10t} = C_1 e^{.10t}$$

$$y = C_1 e^{.10t} - 20{,}000 \qquad \text{But } y(0)=0.$$

$$0 = C_1 e^{0} - 20{,}000 = C_1 - 20{,}000 \Rightarrow$$

$$C_1 = 20{,}000$$

$$y(t) = 20{,}000\, e^{.10t} - 20{,}000$$

$$\boxed{y(t) = 20{,}000\left(e^{.10t} - 1\right)}$$

In particular, after 20 years your balance is

$$y(20) = 20{,}000\left(e^{.10(20)} - 1\right) = \$127{,}781.12$$

$$y = 20\,000(e^{.10t} - 1)$$