Chapter 11 Where we're going...

ex: Consider the differential equation

$$\frac{dy}{dx} = y \qquad \text{subject to} \quad y(0) = 1.$$

Suppose that y is a polynomial:

$$y = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n.$$

When $x = 0$, $y = 1$: $\quad 1 = c_0 + c_1(0) + c_2 \cdot 0^2 + \cdots + c_n 0^n.$

$$\Rightarrow c_0 = 1$$

Equal $\rightarrow$ $y = 1 + c_1 x + c_2 x^2 + \cdots + c_{n-1} x^{n-1} + c_n x^n + \cdots$

Now $\frac{dy}{dx} = y$: $\quad \frac{dy}{dx} = c_1 + 2c_2 x + 3c_3 x^2 + \cdots + n c_n x^{n-1} + (n+1) c_{n+1} x^n + \cdots$

constant coeffs are equal $\Rightarrow$ $c_1 = 1$

x coeffs are equal: $2c_2 = c_1 = 1 \Rightarrow c_2 = \frac{1}{2}$

x^2 coeffs are equal: $3c_3 = c_2 = \frac{1}{2} \Rightarrow c_3 = \frac{1}{3 \cdot 2 \cdot 1}$

x^3 " " " : $4c_4 = c_3 = \frac{1}{3 \cdot 2} \Rightarrow c_4 = \frac{1}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{1}{4!}$

$\vdots$

x^{n-1} " " " " $\quad n c_n = c_{n-1} = \frac{1}{(n-1)!} \Rightarrow c_n = \frac{1}{n \cdot (n-1)!} = \frac{1}{n!}$

BUT, in fact, our function y cannot be a polynomial, because we keep needing to include more terms.

Our solution: $\quad y = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots + \frac{1}{n!}x^n + \cdots$

ex: Same problem by separation of variables

$$\frac{dy}{dx} = y \quad , \quad y(0) = 1$$

$$\int \frac{dy}{y} = \int dx$$

$$\ln|y| = x + C_1$$

$$|y| = e^{x+C_1} = e^{C_1} e^x$$

$$y = \pm e^{C_1} e^x$$

$$y = C e^x$$

$$1 = y(0) = Ce^0 = C \quad \Rightarrow C = 1$$

$$y = e^x$$

Remark: Combining this with the previous example, this suggests

$$y = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

"Power series"

But this brings up other problems, like what is e^1?

$$e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!} + \cdots$$

"Series"

What do we mean by adding infinitely many numbers?

Let S_n = "n^{th} partial sum"

These form the "sequence of partial sums" of the series.

$S_1 = 1$

$S_2 = 1 + 1 = 2$

$S_3 = 1 + 1 + \frac{1}{2} = 2.5$

$S_4 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} = 2.5 + \frac{1}{6} = 2.66\bar{6}$

$S_5 = 2.66\bar{6} + \frac{1}{4!} = 2.66\bar{6} + \frac{1}{24} = 2.7083\bar{3}$

$S_6 = S_5 + \frac{1}{5!} = S_5 + \frac{1}{120} = 2.71\bar{6}$

$\vdots \qquad \vdots$

we are expecting this sequence of numbers to "converges to"

$S = e = 2.71825182 5\ldots$

11.1 Sequences

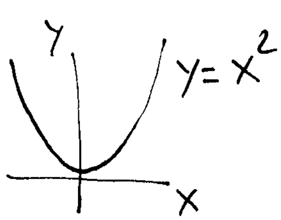
Defn: Informally, a sequence is an infinite list of numbers.
Formally, a sequence is a function whose domain is the positive integers.

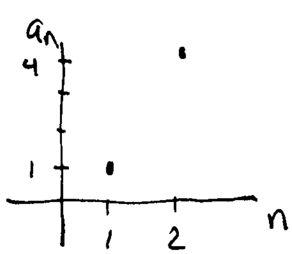
Notation. $f(x) = x^2$ domain $= (-\infty, \infty)$

$a_n = n^2$ domain $= \{1, 2, 3, \cdots\}$

ex. $f(2.5) = (2.5)^2 = 6.25$ or $f(5) = 25$

$a_5 = 5^2 = 25$





Remark: Two typical ways to define a sequence

1. "Explicit" definition

ex: $a_n = n^2$ or $a_n = \frac{n}{n+1}$

$b_n = 2^{n-1}$ so $b_{11} = 2^{11-1} = 2^{10} = 1024$

2. "Recursive" definition

ex: $b_1 = 1$, $b_n = 2b_{n-1}$. "geometric sequence"

k	b_k
1	1
2	2
3	$b_3 = 2b_{3-1} = 2b_2 = 2\cdot 2 = 4$
4	$b_4 = 2b_{4-1} = 2b_3 = 2\cdot 4 = 8$
5	$b_5 = 2b_{5-1} = 2b_4 = 2\cdot 8 = 16$
$\vdots$	

ex: $f_1 = 1$, $f_2 = 1$, $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$.

k	f_k
1	1
2	1
3	2
4	3
5	5
6	8
7	13
8	21
$\vdots$	$\vdots$

ex Geometric sequences

$$a_1 = a \quad , \quad a_n = r a_{n-1} \qquad \text{where } r = \text{"common ratio"}$$

ex. $16, -8, 4, -2, 1, -\frac{1}{2}, \frac{1}{4}, \ldots$ $\qquad a=16$, $r = -1/2$

ex: $3, 6, 12, 24, 48, \ldots$ $\qquad a = 3$, $r = 2$

ex: $3, .3, .03, .003, \ldots$ $\qquad a = 3$, $r = .1$

ex: $a, \ ar, \ ar^2, \ ar^3, \ ar^4$

$\quad \ \, = \quad \, = \quad \ \, = \quad \ = \quad \ =$

$\ \ a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5$ $\qquad \boxed{a_n = ar^{n-1}}$

Definition: $\lim_{n\to\infty} a_n = L$ means that

for any $\varepsilon > 0$ there is an integer N such that

if $n > N$ then $|a_n - L| < \varepsilon$.

[and the demon is defeated.]

ex: $$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{n+1-1}{n+1} = \lim_{n\to\infty} \frac{n+1}{n+1} - \frac{1}{n+1}$$

$$= \lim_{n\to\infty} 1 - \frac{1}{n+1} = 1 - \lim_{n\to\infty} \frac{1}{n+1} = 1 - 0 = 1$$

Easier: If $a_n = \frac{n}{n+1}$ then $a_n = f(n)$ where $f(x) = \frac{x}{x+1}$

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{x+1} \underset{\uparrow}{=} \lim_{x\to\infty} \frac{1}{1} = 1$$

L'Hopital's Rule

Determine whether the sequence converges.

24) $a_n = \dfrac{n^3}{n^3+1} \to 1$ as $n \to \infty$

because $f(x) = \dfrac{x^3}{x^3+1} \to 1$ as $n \to \infty$

26) $a_n = \dfrac{n^3}{n+1}$ $\quad \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{n^3}{n+1} = \infty$ Divergent

ex: $\lim\limits_{n\to\infty} \dfrac{5n^2+2n}{7n^2+3} = \dfrac{5}{7}$

ex. $\lim\limits_{n\to\infty} \dfrac{2n}{3n^3+1} = 0$

30) $\lim\limits_{n\to\infty} \sqrt{\dfrac{n+1}{9n+1}} = \sqrt{\lim\limits_{n\to\infty} \dfrac{n+1}{9n+1}} = \sqrt{\dfrac{1}{9}} = \dfrac{1}{3}$

38) $\lim\limits_{n\to\infty} \dfrac{\ln n}{\ln 2n} = \lim\limits_{n\to\infty} \dfrac{\ln n}{\ln n + \ln 2} \cdot \dfrac{\frac{1}{\ln n}}{\frac{1}{\ln n}}$

$= \lim\limits_{n\to\infty} \dfrac{1}{1 + \frac{\ln 2}{\ln n}} = \dfrac{1}{1+0} = 1$

42) $\lim\limits_{n\to\infty} \ln(n+1) - \ln n = \lim\limits_{n\to\infty} \ln\left(\dfrac{n+1}{n}\right)$

$= \ln\left(\lim\limits_{n\to\infty} \dfrac{n+1}{n}\right) = \ln(1) = 0$

48) $\lim\limits_{n\to\infty} 2^{-n}\cos n\pi = \lim\limits_{n\to\infty} \dfrac{(-1)^n}{2^n} = \lim\limits_{n\to\infty} \left(\dfrac{-1}{2}\right)^n = 0$

<u>Fact</u>: $\lim\limits_{n\to\infty} r^n = 0$ if $|r| < 1$.