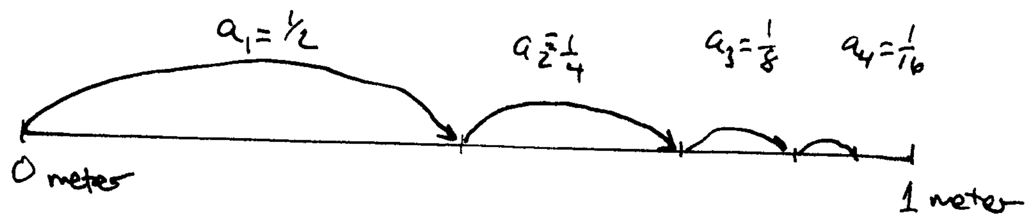
11.2 Series

Informally, is the infinite sum of a sequence of numbers.

ex: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$ "Geometric Series"



Geometric Sequence $\left\{ \begin{array}{l} a_1 = 1/2 \\ a_2 = \frac{1}{4} \\ a_3 = \frac{1}{8} \\ \vdots \\ a_n = \frac{1}{2^n} \\ \vdots \end{array} \right.$

$\left. \begin{array}{l} S_1 = \frac{1}{2} \\ S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \\ S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} \\ \vdots \end{array} \right\}$ Sequence of partial sums

$S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \frac{1}{2^n} = \frac{2^n - 1}{2^n} \to 1$ as $n \to \infty$

a_n = length of a flea's n^{th} jump
= flea's speedometer reading (meters/jump)

S_n = n^{th} partial sum
= location of the flea after n jumps.
= flea's odometer reading (meters)

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{2^n - 1}{2^n} = 1 = S$$

The <u>series</u> $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ converges to $\underline{1}$.

Quick review of geometric series

Geometric sequence: $a, \ ar, \ ar^2, \ ar^3, \ \ldots$

(Infinite) Geometric series:

$$a + ar + ar^2 + ar^3 + \cdots$$

n^{th} partial sum $= S_n = a + ar + ar^2 + \cdots + ar^{n-1}$

Subtract:

$$rS_n = \quad ar + ar^2 + \cdots + ar^{n-1} + ar^n$$

$$S_n - rS_n = a \qquad\qquad -ar^n$$

$$(1-r)S_n = a\,1 - r^n$$

$$\boxed{S_n = a\,\frac{1-r^n}{1-r}}$$

ex: Add the first 10 terms of

$$36 + 12 + 4 + \frac{4}{3} + \frac{4}{9} + \cdots + 36\cdot\left(\frac{1}{3}\right)^9$$

$$= 36 + 12 + 4 + \cdots \qquad + \frac{4}{2187}$$

$$= 36 \cdot \frac{1 - \frac{1}{3^{10}}}{1 - \frac{1}{3}} = 36 \cdot \frac{1 - \frac{1}{3^{10}}}{1-\frac{1}{3}} \cdot \frac{3^{10}}{3^{10}} =$$

$$= 36 \cdot \frac{3^{10} - 1}{3^{10} - 3^9} = 36 \cdot \frac{3^{10}-1}{3^9\cdot(3-1)} = 36 \cdot \frac{59048}{19683(2)}$$

$$= \frac{2}{3^7} \cdot 59048 = \frac{118{,}096}{2{,}187} \qquad = 53.999\,085\,505\,258\,35$$

If $|r| < 1$, the geometric converges to

$$\boxed{S = \frac{a}{1-r}}$$

If $|r| \geq 1$ then series diverges.

Reason: If $|r|<1$,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} a\,\frac{1-r^n}{1-r} = a\cdot\frac{1-0}{1-r} = \frac{a}{1-r}$$

ex: $36 + 12 + 4 + \frac{4}{3} + \frac{4}{9} + \cdots$

Because $r = \frac{1}{3}$ and $|r| = \frac{1}{3} < 1$ the series

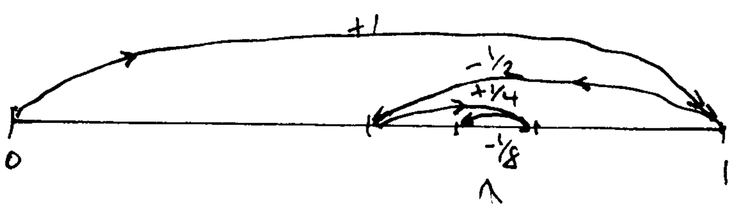
converges to $S = \frac{a}{1-r} = \frac{36}{1-\frac{1}{3}} = \frac{36}{2/3}$

$= 36 \div \frac{2}{3} = 36 \cdot \frac{3}{2} = 54$

ex: $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$, so $a = 1$, $r = -1/2$

Since $|r| = |-1/2| = \frac{1}{2} < 1$ this geometric series is convergent and the sum is

$$S = \frac{a}{1-r} = \frac{1}{1-(-1/2)} = \frac{1}{1+\frac{1}{2}} = \frac{1}{3/2} = \frac{2}{3}$$



$S = .66666$

ex: $a = 1$, $r = 2$.

$$1 + 2 + 4 + 8 + 16 + 32 + 64 + \cdots$$

$$S_n = a \cdot \frac{1-r^n}{1-r} = 1 \cdot \frac{1-2^n}{1-2} = 2^n - 1 \to \infty \text{ as } n \to \infty$$

So this geometric series <u>diverges</u>.

ex: $a = 1$, $r = -1$ $\qquad |r| = 1$

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

$S_1 = 1$ $\qquad$ The series <u>diverges</u>.
$S_2 = 0$
$S_3 = 1$
$S_4 = 0$
$\vdots$

ex: (Harmonic series) $\qquad$ This series is <u>DIVERGENT</u>:

Top flea: $$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \cdots$$

Here's why:

Bottom flea: $$1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}}_{\frac{1}{2}} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}_{\frac{1}{2}} + \underbrace{\frac{1}{16} + \frac{1}{16} + \cdots}_{\frac{1}{2}}$$

Total distance traveled by the bottom flea $= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \cdots$

Since the top flea is always <u>ahead</u> of the bottom flea,
so that the bottom flea travels an infinite distance.

Theorem $\boxed{6}$: If the series $\sum_{n=1}^{\infty} a_n$ is convergent,

then $\lim_{n\to 0} a_n = 0$.

Remark: The converse of this statement is NOT true.

In fact, the harmonic series is a counterexample.

That is, in the harmonic $a_n = \frac{1}{n} \to 0$ as $n \to \infty$

YET $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$.

proof of $\boxed{6}$: Let

$$S_n = a_1 + a_2 + \cdots + a_{n-1} + a_n$$
$$S_{n-1} = a_1 + a_2 + \cdots + a_{n-1}$$

$$S_n - S_{n-1} = a_n$$

Now, if the series converges, then $\lim_{n\to\infty} S_n = S$.

But $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$

$= S - S = 0$ QED.

Logic: [See Math 245: Discrete Math]

(1) If p then q. conditional Statement

(2) If q then p. "converse"

(3) If not q then not p. "contrapositive" is logically equivalent to the original

(4) If not p then not q "inverse" is equivalent to converse.

Contrapositive of $\boxed{6}$: If $\lim_{n\to 0} a_n \neq 0$, then the series $\sum a_n$ is DIVERGENT.

Remark: We'll call this the TEST FOR DIVERGENCE.

Is the series convergent or divergent?

ex: 29) $\sum_{n=1}^{\infty} \frac{n-1}{3n-1} = 0 + \frac{1}{5} + \frac{2}{8} + \frac{3}{11} + \frac{4}{14} + \frac{5}{17} + \cdots$

Observe: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n-1}{3n-1} = \frac{1}{3} \neq 0$.

By the Test for Divergence, the series diverges.

That is the "sequence of partial sums" $= S_n$ diverges.

Remarks: (1) The Test for Divergence is sometimes called "The n^{th}-term test for divergence," or "the n^{th}-term test."

(2) The converse of the Test for Divergence is NOT true.

That is, if $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ may converge

(for instance, $\sum a_n = \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ is a convergent geometric series, for $|r| = \frac{1}{2} < 1$)

OR the series $\sum a_n$ may diverge (for instance, $\sum a_n = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ is the divergent harmonic series).