11.2 Telescoping series

example:

$$
\begin{array}{ccc}
 & & -0^2 \\
1 & 1^2-0^2 & \cancel{1^2-1^2} \\
3 & 2^2-1^2 & \cancel{2^2-2^2} \\
5 & 3^2-2^2 & \cancel{3^2-3^2} \\
7 & 4^2-3^2 & \cancel{4^2-4^2} \\
9 & 5^2-4^2 & 5^2 \\
\hline
 & & 5^2-0^2 = 25
\end{array}
$$

Example 7: Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent and find its sum.
(p707?)

$$\frac{1}{i(i+1)} = \frac{A}{i} + \frac{B}{i+1} = \frac{1}{i} - \frac{1}{i+1}$$

$$1 = A(i+1) + Bi$$

$i=0$: $1 = A$

$i=-1$: $1 = -B \Rightarrow B=-1$

$$n\text{th partial sum} = S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots + \frac{1}{i(i+1)} + \dots + \frac{1}{n(n+1)}$$

$$= \frac{1}{1} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \dots + \cancel{\frac{1}{i}} - \cancel{\frac{1}{i+1}} + \dots + \frac{1}{n} - \frac{1}{n+1}$$

$$= \frac{1}{1} - \frac{1}{n+1} \qquad \lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{n+1} = 1 - 0 = \underline{1}$$

Remark: Does every telescoping series converge? No.

$$\sum_{n=1}^{\infty}(2n-1) = 1+3+5+7+9+\cdots$$

$$S_n = \sum_{i=1}^{n}(2i-1) = 1+3+5+\cdots+(2i-1)+\cdots+(2n-1)$$

$$= (\cancel{1^2}-0^2)+(\cancel{2^2}-\cancel{1^2})+(3^2-\cancel{2^2})+\cdots \quad n^2-(n-1)^2$$

$$= n^2-0^2 \to \infty \text{ as } n\to\infty.$$

Theorem 8: If $\sum a_n$ and $\sum b_n$ are convergent,

then so is $\sum(a_n+b_n)$, and

$$\sum(a_n+b_n) = \sum a_n + \sum b_n$$

ex: $$\sum_{n=1}^{\infty}\frac{2^n-1}{5^n} = \frac{1}{5}+\frac{3}{25}+\frac{7}{125}+\frac{15}{625}+\cdots$$

$$= \sum_{n=1}^{\infty}\left(\frac{2^n}{5^n}-\frac{1}{5^n}\right) = \sum_{n=1}^{\infty}\left(\frac{2}{5}\right)^n - \sum_{n=1}^{\infty}\left(\frac{1}{5}\right)^n$$

(convergent) geometric series

(1) $\sum_{n=1}^{\infty}\left(\frac{2}{5}\right)^n = ?$ $\quad a=\frac{2}{5} \quad r=\frac{2}{5}$

$$S = \frac{a}{1-r} = \frac{\frac{2}{5}}{1-\frac{2}{5}}\cdot\frac{5}{5} = \frac{2}{5-2} = \boxed{\frac{2}{3}}$$

(2) $\sum_{n=1}^{\infty}\left(\frac{1}{5}\right)^n = ?$ $\quad a=\frac{1}{5}, \ r=\frac{1}{5}$

$$S = \frac{a}{1-r} = \frac{\frac{1}{5}\cdot 5}{1-\frac{1}{5}\cdot 5} = \frac{1}{5-1} = \boxed{\frac{1}{4}}$$

So $$\sum_{n=1}^{\infty}\frac{2^n-1}{5^n} = \frac{2}{3}-\frac{1}{4} = \frac{8}{12}-\frac{3}{12} = \boxed{\frac{5}{12}}$$

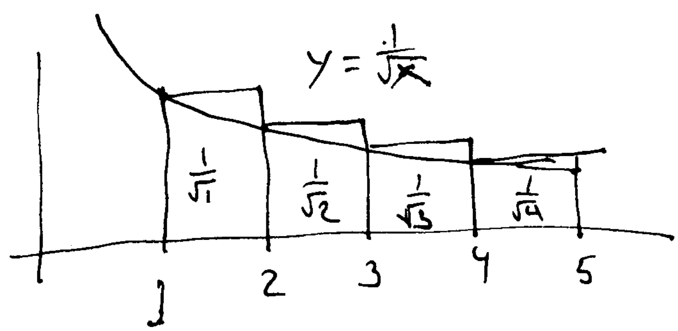
Integral Test : Let f be a continuous, positive, (eventually) decreasing and let $a_n = f(n)$.

(i) If $\int_1^\infty f(x)\,dx$ is convergent, then $\sum_{n=1}^\infty a_n$ is convergent

(ii) If $\int_1^\infty f(x)\,dx$ is divergent, then $\sum_{n=1}^\infty a_n$ is divergent.

ex: Is $\sum_{n=1}^\infty \frac{1}{\sqrt{n}}$ convergent or is it divergent?

$$= \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots$$



Idea:

$$S_4 > \int_1^5 \frac{1}{\sqrt{x}}\,dx$$

But $\int_1^5 x^{-1/2}\,dx = 2x^{1/2}\Big]_1^5$

$$= 2\cdot 5^{1/2} - 2\cdot 1^{1/2}$$

$$= 2\sqrt{5} - 2$$

By picture:

$$S_4 = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} > 2\sqrt{5} - 2$$

$$S_n = \sum_{i=1}^n \frac{1}{\sqrt{i}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \int_1^{n+1} x^{-1/2}\,dx = 2\sqrt{n+1} - 2$$

Because $\int_1^\infty x^{-1/2}\,dx = \lim_{n\to\infty} \int_1^{n+1} x^{-1/2}\,dx = \lim_{n\to\infty} 2\sqrt{n+1} - 2 = \infty$

Conclude $\lim_{n\to\infty} S_n = \infty$.

[Think of a race between a flea and toy car. The flea wins; the car rolls infinite distance $\therefore$ The flea travels an infinite distance.]

P-series Test

The ~~p~~-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p>1$
and divergent if $p \le 1$.

ex: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} = \frac{1}{1} + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \cdots + \frac{1}{n\sqrt{n}} + \cdots$

so $p = 3/2 > 1$. This series converges.

ex: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/2}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} + \cdots$ $p = 1/2$
is divergent.

Harmonic series:
ex: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$ diverges.

Remark: This follows from the Integral Test.

ex: [Integral Test applied to $\sum \frac{1}{n^{3/2}}$.]

$$\int_1^{\infty} \frac{1}{x^{3/2}}\,dx = \int_1^{\infty} x^{-3/2}\,dx \underset{\substack{\uparrow \\ \text{abuse of notation}}}{=} -2x^{-1/2}\Big]_1^{\infty}$$

$$= \frac{-2}{\sqrt{\infty}} + \frac{2}{\sqrt{1}} = -0 + 2 = 2 < \infty$$

$\therefore$ the series converges.

ex: $[p=1]$ $\sum \frac{1}{n}$ diverges:

$$\int_1^{\infty} \frac{1}{x}\,dx = \ln|x|\Big]_1^{\infty} = \ln(\infty) - \ln(1) = \infty$$