11.3 (cont'd) Estimates of Sums using the Integral Test

<u>Notation</u>: If $\sum_{n=1}^{\infty} a_n$ is a convergent series,

then $S_n = \sum_{i=1}^{n} a_i$ = sum of first n terms = n^{th} partial sum

and $\lim_{n\to\infty} S_n = S$. Let R_n = remainder be

the number such $S_n + R_n = S$

add the first 10 (or 50) terms

we'll have some info on the remainder (small)

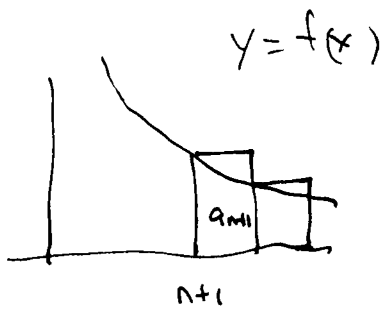
<u>Remainder estimate</u> for the Integral Test:

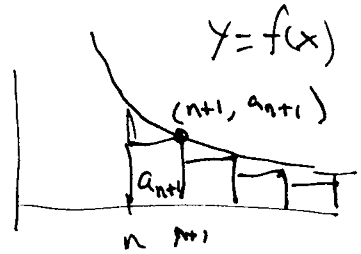
Suppose $f(k) = a_k$ where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ is convergent.

"<u>tail of the series</u>" $= R_n = a_{n+1} + a_{n+2} + a_{n+3} + \dots$

Then $$\boxed{\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx}$$

Explanation by picture:





$R_n = \sum$ area of rectangles $\geq$ area under curve $= \int_{n+1}^{\infty} f(x)\,dx$

$R_n = \sum$ area of rectangles $\leq$ area under curve $= \int_{n}^{\infty} f(x)\,dx$

11.3 36) a) Find s_{10} for $\sum_{n=1}^{\infty} \frac{1}{n^4}$; By calculator, $1.082\ 036\ 583 = s_{10}$

According to Euler, $s = \frac{\pi^4}{90} = 1.082\ 323\ 234$

b) $\int_{11}^{\infty} \frac{1}{x^4}\,dx \le R_{10} \le \int_{10}^{\infty} \frac{1}{x^4}\,dx$

$= \int_{11}^{\infty} x^{-4}\,dx$

$= \left.\frac{x^{-3}}{-3}\right]_{11}^{\infty}$

$= -0 + \frac{11^{-3}}{3} = 2.504 \cdot 10^{-4}$

$= .0002504$

$= \int_{10}^{\infty} x^{-4}\,dx = \left.\frac{x^{-3}}{-3}\right]_{10}^{\infty}$

$= -0 + \frac{10^{-3}}{3}$

$= 3.33 \cdot 10^{-4}$

$= .0003333$

$$s_{10} + \int_{11}^{\infty} \frac{1}{x^4}\,dx \le \overbrace{s_{10} + R_{10}}^{s} \le s_{10} + \int_{10}^{\infty} \frac{1}{x^4}\,dx$$

$$\begin{array}{r} 1.082\ 0366 \\ +\ .000\ 2504 \\ \hline 1.082\ 2870 \end{array} \le s \le \begin{array}{r} 1.082\ 0366 \\ +\ .000\ 3333 \\ \hline 1.082\ 3699 \end{array}$$

This shows that $s = 1.082$ is an estimate of s accurate to three decimal places.

11.4 Comparison Tests

ex: $\sum a_n$ = Series to be studied $= \sum_{n=1}^{\infty} \frac{1}{2^n+1}$

$$= \frac{1}{3} + \frac{1}{5} + \frac{1}{9} + \frac{1}{17} + \frac{1}{33} + \cdots$$

Compare with

$\sum b_n$ = familiar series $= \sum_{n=1}^{\infty} \frac{1}{2^n}$

$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$$

Let $s_n = \sum_{k=1}^{n} a_k$ = flea "a"s location after n jumps

$t_n = \sum_{k=1}^{n} b_k$ = flea "b's" " " "

Since $a_k < b_k$ for every k, $s_n < t_n$, so flea b is ahead.

BUT $\sum b_n = \sum \left(\frac{1}{2}\right)^n$ is a geometric series with $r = \frac{1}{2}$ so it converges.

That is $\lim_{n\to\infty} t_n = t = \frac{a}{1-r} = \frac{1}{2} \cdot \frac{1}{1-\frac{1}{2}} = \frac{1}{2-1} = 1$

Since s_n is a monotically increasing sequence bounded above (by 1), it follows that $\sum a_n = \sum \frac{1}{2^n+1}$ converges.

Remark: We have just applied the Direct Comparison Test.

Theorem: "Direct Comparison Test"

Suppose that $\sum a_n$ = series with positive terms (to be studied)

and $\sum b_n$ = series with positive terms (familiar series)

(i) If $\sum b_n$ converges and $a_n < b_n$ for all n then $\sum a_n$ converges.

(ii) If $\sum b_n$ diverges and $a_n > b_n$ for all n then $\sum a_n$ diverges.

Remark: In practice we will pick $\sum b_n$ to be either a p-series or a geometric series.

8) $$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{6^n}{5^n - 1}$$

Observe: $$5^n - 1 < 5^n$$

$$\frac{1}{5^n - 1} > \frac{1}{5^n}$$

$$a_n = \frac{6^n}{5^n - 1} > \frac{6^n}{5^n} = \left(\frac{6}{5}\right)^n = b_n$$

Let's take $b_n = \sum_{n=1}^{\infty} \left(\frac{6}{5}\right)^n$ a divergent geometric series because $|r| = \left|\frac{6}{5}\right| > 1$.

$\therefore$ $\sum a_n$ diverges, since $a_n > b_n$ for all n, and $\sum b_n$ diverges.

Limit Comparison Test:

$\sum a_n$ = series to be studied $\qquad$ $\sum b_n$ = familiar series chosen so that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = C$$

where $0 < C < \infty$.

Then if $\sum b_n$ converges, $\sum a_n$ converges

if $\sum b_n$ diverges, $\sum a_n$ diverges.

14) $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n-1} = \sum a_n$

How to choose $\sum b_n$? Try $b_n = \frac{\sqrt{n}}{n} = \frac{n^{1/2}}{n} = \frac{1}{n^{1/2}}$

so $\sum b_n = \displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{1/2}}$ is a p-series with $p = 1/2$ so is divergent.

$$\frac{a_n}{b_n} = a_n \cdot \frac{1}{b_n} = \frac{n^{1/2}}{n-1} \cdot \frac{n^{1/2}}{1} = \frac{n}{n-1}$$

$\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n-1} = 1$ and $0 < 1 < \infty$.

$\therefore$ So both $\sum a_n$ and $\sum b_n$ diverge.

Some examples added after class:

more 11.4: Convergence Tests.

Determine whether $\sum a_n$ converges:

example: $\displaystyle \sum a_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)(n+2)}}$

Because $\displaystyle a_n = \frac{1}{\sqrt{n(n+1)(n+2)}} = \frac{1}{\sqrt{n^3+3n^2+2n}}$

try taking $\displaystyle b_n = \frac{1}{\sqrt{n^3}} = \frac{1}{n^{3/2}}$. Then $\displaystyle \sum b_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series $(p = \frac{3}{2} > 1)$.

Apply the Limit Comparison Test:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt{n^3+3n^2+2n}} \cdot \frac{\sqrt{n^3}}{1} = \lim_{n\to\infty} \sqrt{\frac{n^3}{n^3+3n^2+2n}}$$

$$= \sqrt{\lim_{n\to\infty} \frac{n^3}{n^3+3n^2+2n} \cdot \frac{1/n^3}{1/n^3}}$$

$$= \sqrt{\lim_{n\to\infty} \frac{1}{1+3/n+2/n^2}} = \sqrt{\frac{1}{1+0+0}} = 1 > 0$$

Since $0 < 1 < \infty$, conclude that BOTH $\sum a_n$ and $\sum b_n$ $\boxed{\text{CONVERGE!}}$

ex: $\displaystyle \sum a_n = \sum_{n=1}^{\infty} \frac{3n+5}{n2^n}$ When n is large, 2^n is dominant, so try

$\displaystyle b_n = \frac{3n}{n2^n} = \frac{3}{2^n}$ or more simply take $b_n = \frac{1}{2^n}$.

Then $\displaystyle \sum b_n = \sum_{n=1}^{\infty} \frac{1}{2^n}$ is a convergent geometric series $\left(|r| = \left|\frac{1}{2}\right| < 1\right)$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{3n+5}{n2^n} \cdot \frac{2^n}{1} = \lim_{n\to\infty} \frac{3n+5}{n} = 3.$$

Since 3 is finite and $3>0$, BOTH $\sum a_n$ and $\sum b_n$ $\boxed{\text{converge.}}$

ex: $\sum a_n = \sum_{n=1}^{\infty} \tan\left(\frac{1}{n}\right)$ Take $b_n = \frac{1}{n}$

Then $\sum b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent p-series (i.e. the harmonic series, $p=1$)

$$\frac{a_n}{b_n} = a_n \cdot \frac{1}{b_n} = \tan\left(\frac{1}{n}\right) \cdot \frac{n}{1} = \frac{n\sin(1/n)}{\cos(1/n)} = \frac{\sin(1/n)}{(1/n)} \cdot \frac{1}{\cos(1/n)}$$

Recall that $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ and $\lim_{\theta \to 0} \cos\theta = 1$

So $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} \cdot \lim_{n\to\infty} \frac{1}{\cos(1/n)} = 1 \cdot \frac{1}{1} = 1 > 0$

$\therefore$ Both $\sum a_n$ and $\sum b_n$ are $\boxed{\text{DIVERGENT}}$.

ex: $\sum a_n = \sum_{n=1}^{\infty} \frac{n + \ln n}{n^2+1}$ when n is large n dominates $\ln n$ and n^2 dominates 1.

This suggests letting $b_n = \frac{n}{n^2} = \frac{1}{n}$

$\sum b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is the divergent harmonic series.

$$\frac{a_n}{b_n} = \frac{n + \ln n}{n^2+1} \cdot \frac{n}{1} = \frac{n^2 + n\ln n}{n^2+1} \cdot \frac{\frac{1}{n^2}}{1/n^2} = \frac{1 + \frac{\ln n}{n}}{1 + 1/n^2}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1 + \frac{\ln n}{n}}{1 + 1/n^2} = \frac{1+0}{1+0} = 1 > 0$$

By the LIMIT COMPARISON TEST, both $\sum a_n$ and $\sum b_n$ $\boxed{\text{DIVERGE.}}$