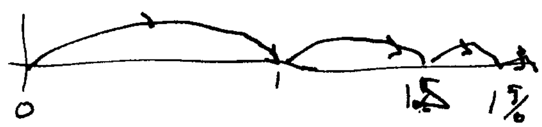
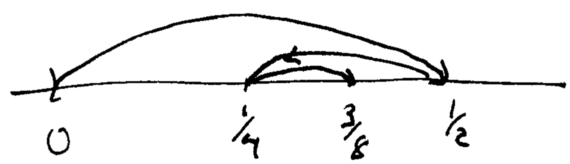
# 11.5 Alternating Series

**Remark:** Not all series are positive term series.

ex: $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ is a positive term series.



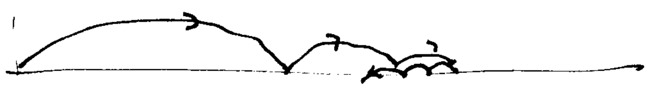

ex: $\sum_{n=1}^{\infty} -\left(\frac{-1}{2}\right)^n$ is an alternating series




$$= \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots$$

ex: $\sum_{n=1}^{\infty} \frac{\sin(n)}{n} = .841 + .455 + .047 - .190$

$$- .192 - .047 + .094 + .124 + \cdots$$



**Alternating Series Test:** If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots \qquad b_n \geq 0$$

satisfies (i) $b_{n+1} \leq b_n$ for all $n$

(ii) $\lim_{n\to\infty} b_n = 0$

then the series converges.

ex: Alternating Harmonic Series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

that is $b_1 = 1$, $b_2 = 1/2$, $b_3 = 1/3$, $b_4 = 1/4$, $b_5 = 1/5$, $\vdots$

Do conditions (i) and (ii) hold?

(i) $b_{n+1} \le b_n$ for all $n$?

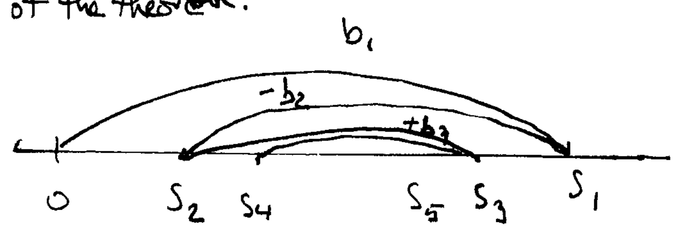
Yes $\frac{1}{n+1} < \frac{1}{n}$ for all $n$.

(ii) $\lim_{n\to\infty} b_n = 0$ ?

Yes $\lim_{n\to\infty} \frac{1}{n} = 0$.

Conclusion: The alternating harmonic series converges.

Idea of the proof of the theorem.




when $n$ is even, the sequence $S_2, S_4, S_6, S_8 \ldots$

is a monotonically increasing sequence (because $b_{n+1} < b_n$)

bounded above by $b_1$ hence converges to a number $S$.

When $n$ is odd, $S_1, S_3, S_5 \ldots$ is a decreasing sequence

and

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} (S_{2n} + b_{2n+1})$$
$$= \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} b_{2n+1}$$
$$= S + 0 = S$$

Since both $\{S_{2n}\}$ and $\{S_{2n+1}\}$ converge to $S$, so does $\{S_n\}$.

11.5 #4) Test the series for convergence or divergence

$$\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}}$$ Is this alternating? Yes.

(i) $b_{n+1} \leq b_n$ ?

$$n+1 > n$$
$$\sqrt{n+1} > \sqrt{n}$$
$$\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$$

<u>OR</u>: Consider $f(x) = \frac{1}{\sqrt{x}} = x^{-1/2}$

$$f'(x) = -\frac{1}{2}x^{-3/2} = \frac{-1}{2\sqrt{x^3}}$$

and if $x>0$, $f'(x) < 0$ hence decreasing.

So $b_n = f(n+1)$ is a decreasing <u>sequence</u>.

(ii) $\lim_{n\to\infty} b_n = 0$ ? $\lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0$.

$\therefore$ The series converges.

#7) $\sum_{n=1}^{\infty} (-1)^n \frac{3n-1}{2n+1} = -\frac{2}{3} + \frac{5}{5} - \frac{8}{7} + \frac{11}{9} - \cdots$

(i) $b_{n+1} \leq b_n$? No. (ii) $\lim_{n\to\infty} b_n = 0$? No

$$\lim_{n\to\infty} b_n = \frac{3}{2} \neq 0$$

Since $\lim_{n\to\infty} (-1)^n b_n$ does not exist, by the TEST for DIVERGENCE the series diverges.

20) $\sum_{n=1}^{\infty} (-1)^n \left( \sqrt{n+1} - \sqrt{n} \right)$

ii) $\lim_{n\to\infty} \left( \sqrt{n+1} - \sqrt{n} \right) = ?$

$$\frac{\sqrt{n+1} - \sqrt{n}}{1} \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}}$$

$$= \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0$$

i) $b_{n+1} \leq b_n$ ? Consider $f(x) = \sqrt{x+1} - \sqrt{x}$
$= (x+1)^{1/2} - x^{1/2}$

$$f'(x) = \frac{1}{2}(x+1)^{-1/2} - \frac{1}{2}\left(x^{-1/2}\right)$$

$$= \frac{1}{2}\left[ \frac{1}{\sqrt{x+1}} - \frac{1}{\sqrt{x}} \right]$$

$$= \frac{1}{2}\left[ \frac{\sqrt{x} - \sqrt{x+1}}{\sqrt{x(x+1)}} \right]$$

Is $\sqrt{x} - \sqrt{x+1} < 0$ ? Assume $x > 0$.

$-1 = x - (x+1) < 0$ is true.

multiply both sides by $\frac{1}{\sqrt{x} + \sqrt{x+1}}$ ← positive if $x > 0$.

Then $\frac{x - (x+1)}{\sqrt{x} + \sqrt{x+1}} < 0$

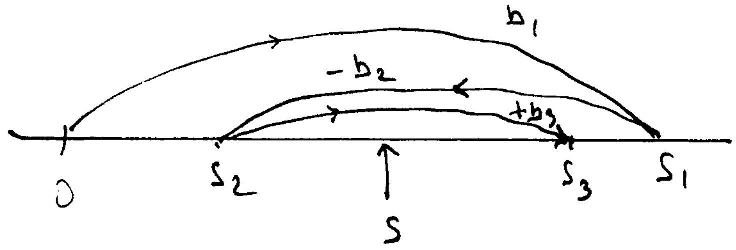
$$\sqrt{x} - \sqrt{x+1} = \frac{\left(\sqrt{x} - \sqrt{x+1}\right)\cancel{\left(\sqrt{x} + \sqrt{x+1}\right)}}{\cancel{\sqrt{x} + \sqrt{x+1}}} < 0$$

so $\sqrt{x} - \sqrt{x+1} < 0$
so $f'(x) < 0$
so $b_{n+1} \leq b_n$.

∴ This alternating series converges, by the Alternating Series Test.

## Alternating series estimation theorem

Assume the Alternating Series Test applies.




Let $|R_n| = |s - s_n|$

then $|R_n| = |s - s_n| \le b_{n+1}$

In words: The distance between $s_n$ and $s$ is smaller than $b_{n+1}$ = the first term not included in $s_n$.

[Added after class →]

Reason: By the proof of the Alternating Series Test, the sum $s$ lies between the partial sums $s_n$ and $s_{n+1}$. It follows that $|s - s_n| \le |s_{n+1} - s_n| = b_{n+1}$.

example [§11.5 #24]: Suppose we want to estimate the sum of the convergent alternating series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n5^n}$ with $|\text{error}| < 0.0001 = \frac{1}{10,000} = 10^{-4}$. How many terms are needed? What is the estimate of the sum?

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n5^n} = \frac{-1}{5} + \frac{1}{2\cdot5^2} - \frac{1}{3\cdot5^3} + \frac{1}{4\cdot5^4} - \frac{1}{5\cdot5^5} + \frac{1}{6\cdot5^6} - \frac{1}{7\cdot5^7} + \cdots$$

$$= \frac{-1}{5} + \frac{1}{50} - \frac{1}{375} + \frac{1}{2500} - \frac{1}{15\,625} + \frac{1}{93\,750} - \frac{1}{546\,875} + \cdots$$

If we approximate $s$ by $s_4 = -\frac{1}{5} + \frac{1}{50} - \frac{1}{375} + \frac{1}{2500}$

$$= \frac{9}{50} - \frac{1}{375} + \frac{1}{2500} = -\frac{137}{750} + \frac{1}{2500}$$

$$= -\frac{1367}{7500} = -0.182\,266\bar{6}$$

Then $s_4 - s \le b_5 = \frac{1}{15\,625} = 6.4\cdot10^{-5} < 10^{-4} = 0.0001$.

[A better approximation is $s_{16} = -0.182\,321\,556\,794$ for $b_{17} = 7.7\cdot10^{-14}$.]