11.6 Absolute Convergence

Defn: A series $\sum a_n$ (where some a_n's are positive and some are negative) is absolutely convergent if $\sum |a_n|$ is convergent.

ex: $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \cdots$$
$$= 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \cdots$$

Is this series absolutely convergent?

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$

Is this series, $\sum |a_n|$, convergent? Yes,

$\sum |a_n|$ is a p-series with $p=2$, so it converges.

That means the series $\sum \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$ converges absolutely.

ex: $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

This is "the alternating harmonic series". Last time, we showed this converges by the A.S.T. BUT, does this series converge absolutely?

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$$

Does this series converge? No its the divergent harmonic series.

So the original series converges but not absolutely. It converges conditionally.

<u>Theorem</u>: If a series $\sum a_n$ has the property that

$\sum |a_n|$ converges, then $\sum a_n$ converges.

That is, if $\sum a_n$ converges <u>absolutely</u> then it converges.

<u>Remark</u>: There are exactly <u>three possibilities</u>: (1) $\sum |a_n|$ converges, so also $\sum a_n$ converges.

That is, $\sum a_n$ is <u>absolutely convergent</u>.

(2) $\sum |a_n|$ diverges, but $\sum a_n$ converges,

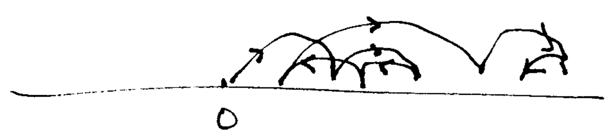
That is, $\sum a_n$ is <u>conditionally convergent</u>.

(3) $\sum a_n$ diverges so also $\sum |a_n|$ will diverge,

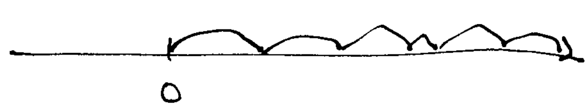
That is, $\sum a_n$ is <u>divergent</u>.

proof (via the story of three fleas):

$s_n =$ papa flea's location $= \sum_{k=1}^{n} a_k$

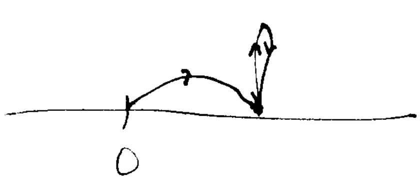


$t_n =$ mama flea's location $= \sum_{k=1}^{n} b_k = \sum_{k=1}^{n} |a_k|$



<u>Assume</u> mama flea reaches a limit: $\lim_{n\to\infty} \sum |a| = L < \infty$.

$u_n =$ baby flea's location: $\sum_{k=1}^{n} c_k = \sum_{k=1}^{n} \left(\frac{|a_k| + a_k}{2}\right)$ ← positive term series



Note: If $a_k \geq 0$, $\frac{|a_k| + a_k}{2} = \frac{a_k + a_k}{2} = a_k$

If $a_k < 0$, $\frac{|a_k| + a_k}{2} = \frac{-a_k + a_k}{2} = 0$

Mama flea's location $= t_n = \sum_{k=1}^{n} b_k$ is monotonically increasing sequence $\le L$.

baby flea's location $= u_n = \sum_{k=1}^{n} c_k$ is a monotonically increasing sequence, but each $c_k \le b_k$

So u_n is also a monotically increasing sequence, bounded by L.

So $\lim_{n\to\infty} u_n = \sum^{\infty} c_k = M \le L$

Remark: One can show that u_n is the midpoint between s_n and t_n for every n.

$$\frac{|a_k| + a_k}{2} = \frac{|a_k|}{2} + \frac{a_k}{2} \quad \leftarrow \text{ solve this for } a_k$$

$$\frac{|a_k| + a_k}{2} - \frac{|a_k|}{2} = \frac{a_k}{2}$$

$$2\left(\frac{|a_k| + a_k}{2}\right) - |a_k| = a_k$$

$$2\,c_k \quad - b_k = a_k$$

Since $\sum_{k=1}^{\infty} a_k = 2\underbrace{\sum_{k=1}^{\infty} c_k}_{\text{convergent}} - \underbrace{\sum_{k=1}^{\infty} b_k}_{\text{convergent}} = 2M - L$

Because the sum & difference of a convergent series is convergent, (papa flea settles down) $\sum_{k=1}^{\infty} a_k$ is convergent.

Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

4) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+4}$

Does the series converge absolutely? $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+4} = \sum a_n$

Do limit comparison test with $\displaystyle\sum b_n = \sum \frac{n}{n^2} = \sum \frac{1}{n} \leftarrow$ Divergent

and $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n^2+4} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n^2}{n^2+4} = 1$

and $0<1<\infty$.

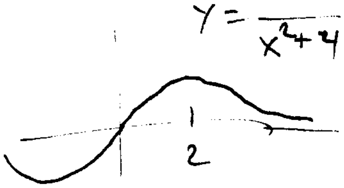
$\therefore \displaystyle\sum (-1)^{n-1} \frac{n}{n^2+4}$ does NOT converge absolutely.

Does it converge at all? Apply the alternating series test.

$b_n = \dfrac{n}{n^2+4}$ i) $b_{n+1} \overset{?}{\leq} b_n$

Let $f(x) = \dfrac{x}{x^2+4}$. Then $f'(x) = \dfrac{1\cdot(x^2+4) - x(2x)}{(x^2+4)^2}$

$y = \dfrac{x}{x^2+4}$



$= \dfrac{-x^2+4}{(x^2+4)^2} < 0$

provided $-x^2+4<0$

or if $4<x^2$

or if $2<x$.

ii) $\displaystyle\lim_{n\to\infty} \frac{n}{n^2+4} = 0$ yes.

$\therefore$ The series is conditionally convergent.

ex: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{n+1} = \frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \frac{5}{6} \cdots$

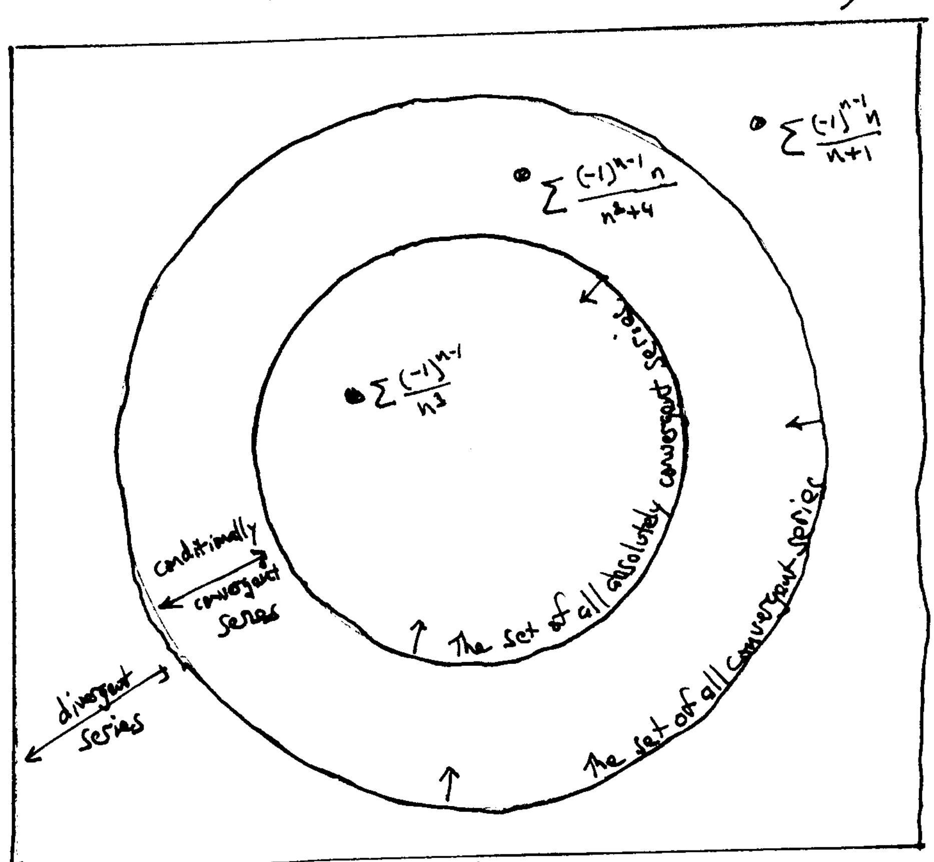
Since $\displaystyle\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$, then $\displaystyle\lim_{n\to\infty} \frac{(-1)^{n-1} n}{n+1}$ does not exist,

So the series diverges by the Test for Divergence.

ex: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} = 1 - \frac{1}{8} + \frac{1}{27} - \frac{1}{64} + \cdots$

Because $\sum \frac{1}{n^3}$ converges (p-series with $p=3$)

the series converges absolutely. (So we need not bother with the A.S.T.)



The set of all series