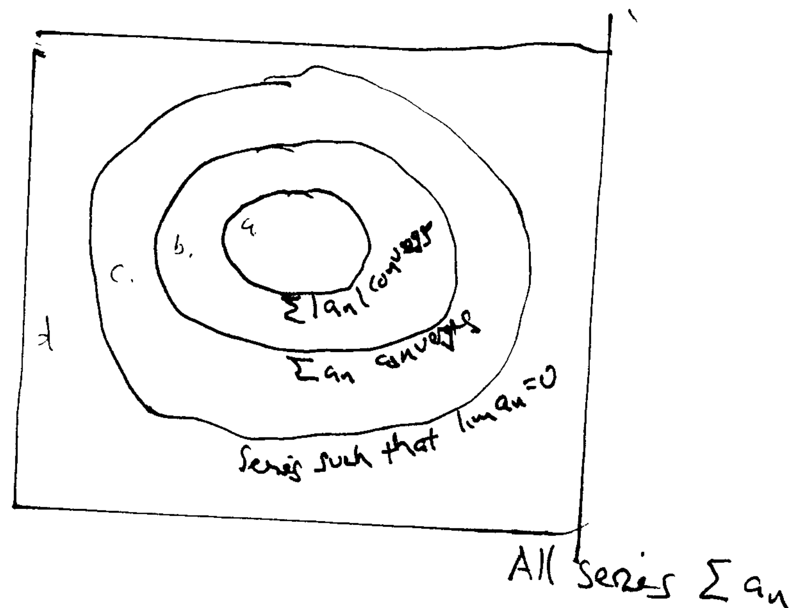
11.6 Ratio and Root Tests

from last time



$\sum |a_n|$ converges $\Rightarrow$ $\sum a_n$ converges $\Rightarrow$ ~~$\not$~~ $\lim_{n\to\infty} a_n = 0$

i.e. $\sum a_n$ is absolutely convergent — i.e. $\sum a_n$ is convergent — i.e. the terms of the series tend toward zero.

<u>Examples</u>: a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ converges absolutely

b) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges conditionally

c) $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, yet $\lim_{n\to\infty} \frac{1}{n} = 0$

d) $\sum_{n=1}^{\infty} \frac{n}{n+1}$ $\qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} \neq 0$

<u>Ratio Test</u>: For $\sum a_n$

(i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$ then $\sum a_n$ converges absolutely.

(ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} L > 1 \\ \text{or} \\ \infty \end{cases}$ then $\sum a_n$ diverges.

(iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$ you gotta find a different convergence test.

<u>ex</u>: $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-3)^{n-1} 5^n}{n!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-3)^n 5^{n+1}}{(n+1)!} \cdot \frac{n!}{(-3)^{n-1} 5^n}\right|$$

$$= \frac{3^n}{3^{n-1}} \cdot \frac{5^{n+1}}{5^n} \cdot \frac{n!}{(n+1)!} = \frac{3}{1} \cdot \frac{5}{1} \cdot \frac{1}{n+1}$$

<u>Remark</u>: $5! = 5\cdot4\cdot3\cdot2\cdot1 = 5\cdot(4\cdot3\cdot2\cdot1) = 5\cdot4!$

So $\frac{5!}{4!} = \frac{5\cdot4!}{4!} = 5$. More generally: $\frac{(n+1)!}{n!} = \frac{(n+1)\cancel{n!}}{\cancel{n!}} = n+1$

ex (cont'd): $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{3\cdot5}{n+1} = \lim_{n\to\infty} \frac{15}{n+1} = 0 < 1$

$\therefore \sum a_n$ converges absolutely.

14) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{10}}{(-10)^{n+1}} = \sum a_n$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^{10}}{(-10)^{n+2}} \cdot \frac{(-10)^{n+1}}{n^{10}}\right| = \frac{(n+1)^{10}}{n^{10}} \cdot \frac{10^{n+1}}{10^{n+2}}$$

$$= \left(\frac{n+1}{n}\right)^{10} \cdot \frac{1}{10} \to 1^{10} \cdot \frac{1}{10} = \frac{1}{10} < 1 \quad \text{as } n \to \infty$$

$\therefore$ $\sum a_n$ converges absolutely.

ex: $\displaystyle\sum_{n=0}^{\infty} \frac{(2n+1)!}{3^n} = \sum a_n$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(2(n+1)+1)!}{3^{n+1}} \cdot \frac{3^n}{(2n+1)!}$$

$$= \frac{(2n+3)!}{(2n+1)!} \cdot \frac{3^n}{3^{n+1}} = \frac{(2n+3)\cdot(2n+2)\cdot\cancel{(2n+1)!}}{\cancel{(2n+1)!}} \cdot \frac{3^n}{3^{n+1}}$$

$$= \frac{(2n+3)(2n+2)}{3} = \frac{4n^2+10n+6}{3} \to \infty \quad \text{as } n \to \infty$$

$\therefore$ $\sum a_n$ diverges.

ex: 4) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+4}$ TRY ratio test:

converge absolutely? conditionally? diverge?

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{(n+1)^2+4} \cdot \frac{n^2+4}{n} = \frac{n^3+\cdots}{n^3+\cdots} \to 1 \quad \text{as } n \to \infty$$

So the Ratio Test tells us nothing. Apply some other test.

4 cont'd) Does $\sum a_n$ converge absolutely? No.

$\sum \tilde{a}_n = \sum \frac{n}{n^2+4}$ let $\sum b_n = \sum \frac{n}{n^2} = \sum \frac{1}{n}$

← divergent Harmonic series

$$\lim_{n\to\infty} \frac{\tilde{a}_n}{b_n} = \lim \frac{n}{n^2+4} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n^2}{n^2+4} = 1 > 0$$

$\therefore$ BOTH series diverge.

Does the original series converge conditionally?

Apply A.S.T:

i) $b_{n+1} < b_n$ let $f(x) = \frac{x}{x^2+4}$

$$f'(x) = \frac{1\cdot(x^2+4) - x(2x)}{(x^2+4)^2} = \frac{-x^2+4}{(x^2+4)^2}$$

So $f'(x) < 0$ provided $x > 2$

So $b_{n+1} < b_n$ provided $n > 2$.

ii) $\lim_{n\to\infty} \frac{n}{n^2+4} = 0$

$\therefore$ The alternating series $\sum (-1)^{n-1} \frac{n}{n^2+4}$ converges

$\therefore$ So converges conditionally.

Root Test: For $\sum a_n$

i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$ then $\sum a_n$ converges absolutely

ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \begin{cases} L > 1 \\ \text{or} \\ \infty \end{cases}$ then $\sum a_n$ diverges

iii) If $\lim_{n\to} \sqrt[n]{|a_n|} = 1$ look for some other test.

22) $\displaystyle\sum_{n=2}^{\infty} \left(\frac{-2n}{n+1}\right)^{5n}$

$$\sqrt[n]{|a_n|} = |a_n|^{1/n} = \left[\left(\frac{2n}{n+1}\right)^{5n}\right]^{1/n} = \left(\frac{2n}{n+1}\right)^{5}$$

$$= \frac{32\,n^5}{(n+1)^5} \longrightarrow 32 > 1 \text{ as } n \to \infty \qquad \text{DIVERGENT}$$

11.8 Power Series

What is a power series? Informally, a polynomial that doesn't know when to stop.

examples: $1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$

$$1 - (x-2) + \frac{(x-2)^2}{2} - \frac{(x-2)^3}{3} + \frac{(x-2)^4}{4} - \cdots$$

A "power series centered at $a=2$"

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots = \sin x$$

Bessel function of order 0 $= \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} = 1 - \frac{1}{2^2(1!)^2}x^2 + \frac{1}{2^4(2!)^2}x^4 - \cdots$

<u>Remark</u>: Every polynomial is a power series for which all but a finite number of coefficients are zero.

Remark: Any polynomial can be consider to be a polynomial centered at $a=2$.

$$P(x) = 3x^3 + x^2 + 4x + 2 = (x-2)(3x^2+7x+18)+38$$

$$= (x-2)\left[(x-2)(3x+13)+44\right] + 38$$

$$= (x-2)\left[(x-2)\left((x-2)3+19\right)+44\right]+38$$

$$= 3(x-2)^3 + 19(x-2)^2 + 44(x-2) + 38$$

divide by $x-2$ $\longrightarrow$

$$\begin{array}{r|rrrr}
2 & 3 & 1 & 4 & 2 \\
 & & 6 & 14 & 36 \\
\hline
 & 3 & 7 & 18 & \vert\ 38 \\
 & & 6 & 26 & \\
\hline
 & 3 & 13 & \vert\ 44 & \\
 & & 6 & & \\
\hline
 & 3 & \vert\ 19 & &
\end{array}$$

[To be completed after class]

That is, $P(x)$ can both be thought of as a powerseries centered at $a=0$:

$$P(x) = 2 + 4x + x^2 + 3x^3 + 0x^4 + 0x^5 + \cdots$$

OR as a power series centered at $a=2$:

$$P(x) = 38 + 44(x-2) + 19(x-2)^2 + 3(x-2)^3 + 0(x-2)^4 + 0(x-2)^5 + \cdots$$