11.8 Power Series (cont'd)

Remarks: Two ways to think about power series:

1. As a polynomial of "infinite degree".
2. As a family of series, one for each x.

Remark:

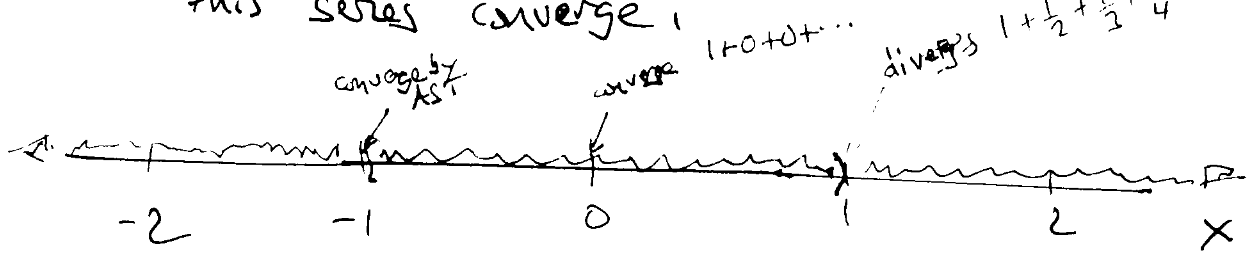
$1 + \frac{1}{2} + \frac{1}{4}$ finite sum: can be generalized two ways

finite sum; infinitely many values of x: $1 + x + x^2$

infinite sum: $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$

$1 + x + x^2 + x^3 + \cdots$ infinite sum, infinitely many values for x.

ex: For the power series $1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \cdots$ we have a different for each possible value of x. Some will converge some will diverge. For which values of x will this series converge?



We've got to be more effecient about testing the convergence for various values of x.

ex (cont'd) $1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n+1}$

Apply the Ratio Test where we think of $x = \underline{\text{constant}}$.

$a_n = \frac{x^n}{n+1}$ so $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{n+2} \cdot \frac{n+1}{x^n}\right|$

$$= \frac{n+1}{n+2} \cdot \left|\frac{x^{n+1}}{x^n}\right|$$

$$= \frac{n+1}{n+2} \cdot |x|$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n+1}{n+2} \cdot |x| = \frac{1}{1} \cdot |x| = |x|$$

So, by the Ratio Test, does this series converge? It depends.

If $|x| < 1$ the series converges absolutely.

If $|x| > 1$ the series diverges.

If $|x| = 1$, that is, if $x = 1$ or $x = -1$, the test fails

The "interval of convergence" is $[-1, 1)$.

the "radius of convergence" is 1

<u>Remark</u>: <u>One</u> power series is really infinitely many series, one for each x.

Theorem: For the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ there are only three possibilities:

(i) the series converges only for $x=a$. $R=0$

(ii) the series converges for all reals. $R=\infty$

(iii) the series converges if $|x-a|<R$ and diverges if $|x-a|>R$, for $0<R<\infty$

[R = radius of convergence.]

examples: Find the radius and interval of convergence.

4) $$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt[3]{n}} = -x + \frac{x^2}{\sqrt[3]{2}} - \frac{x^3}{\sqrt[3]{3}} + \frac{x^4}{\sqrt[3]{4}} - \cdots$$

Ratio test: $$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{\sqrt[3]{n+1}} \cdot \frac{\sqrt[3]{n}}{x^n}\right| = \sqrt[3]{\frac{n}{n+1}}\,|x|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \sqrt[3]{\frac{n}{n+1}}\,|x| = |x| \sqrt[3]{\lim_{n\to\infty}\frac{n}{n+1}} = |x|$$

So the series converges if $|x|<1$ so $-1<x<1$.

" " diverges if $|x|>1$ that is $x<-1$ or $x>1$.

$\therefore$ The radius of converge = 1. Interval of convergence?

The Ratio test failed for $|x|=1$, i.e. $x=1$ or $x=-1$.

Consider $x=1$: $-1 + \frac{1}{\sqrt[3]{2}} - \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$ will converge by the A.S.T. (details omitted)

Consider $x=-1$: $\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{\sqrt[3]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$, a divergent p-series.

Interval of convergence = $(-1, 1]$

8) $\displaystyle\sum_{n=1}^{\infty} n^n x^n = x + 2^2x^2 + 3^3x^3 + 4^4x^4$ Apply Ratio Test:

[7th Ed.]

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^{n+1}\ x^{n+1}}{n^n\ x^n}\right| = \frac{(n+1)^{n+1}}{n^n}\cdot|x|$$

$$= \frac{(n+1)^n}{n^n}\cdot(n+1)\,|x| = \left(\frac{n+1}{n}\right)^n (n+1)|x|$$

$$= \left(1+\frac{1}{n}\right)^n (n+1)|x| \qquad \underline{\text{Fact}}: \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x|\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\cdot(n+1) = \begin{cases}\infty & \text{if } |x|\neq 0\\ 0 & \text{if } |x| = 0\end{cases}$$

The series converges only if $x=0$; so $R=0$.

Easier: Use the Root Test.

$$\sqrt[n]{|a_n|} = \left(n^n x^n\right)^{1/n} = (n^n)^{1/n}\ |x|^{n\cdot 1/n}$$
$$= n|x|$$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty} n|x| = \begin{cases}\infty & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$$

"Interval of convergence" $= \{0\}$

18) $\displaystyle\sum_{n=1}^{\infty}\frac{n}{4^n}(x+1)^n$

[7th ed]

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)\ (x+1)^{n+1}}{4^{n+1}}\cdot\frac{4^n}{n\,(x+1)^n}\right| = \frac{n+1}{n}\cdot\frac{1}{4}|x+1|$$

$$\lim_{n\to\infty}\frac{n+1}{n}\cdot\frac{1}{4}|x+1| = \frac{|x+1|}{4}$$

18 cont'd) <u>converges</u> if $\frac{|x+1|}{4} < 1$

So $|x+1| < 4$ $\Rightarrow$ Radius of conv. is 4

$$-4 < x+1 < 4$$
$$-1 \quad -1 \quad -1$$

Converges for these x: $-5 < x < 3$ Diverges if $|x+1| > 4$.

What's left? $|x+1| = 4$ i.e. $x = 3$ or $x = -5$.

[Completed after class:]

At $x = 3$, the series is $\sum_{n=1}^{\infty} \frac{n}{4^n}(3+1)^n = \sum_{n=1}^{\infty} \frac{n\,4^n}{4^n} = \sum_{n=1}^{\infty} n$

$= 1+2+3+4+5+\cdots$

which diverges by the Test for Divergence.

At $x = -5$, the series is $\sum_{n=1}^{\infty} \frac{n}{4^n}(-5+1)^n = \sum_{n=1}^{\infty} \frac{n(-4)^n}{4^n} = \sum_{n=1}^{\infty} (-1)^n n$

$= -1+2-3+4-5+\cdots$

which also diverges by the Test for Divergence.

$\therefore$ The <u>interval</u> of convergence is $(-5, 3)$.

Find the radius and interval of convergence of

27) [8th ed]

$$\sum_{n=1}^{\infty} \frac{x^n}{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)} = x + \frac{x^2}{3} + \frac{x^3}{3\cdot 5} + \frac{x^4}{3\cdot 5\cdot 7} + \frac{x^5}{3\cdot 5\cdot 7\cdot 9} + \cdots$$

Apply the Ratio Test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)(2n+1)} \cdot \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}{x^n}\right|$$

$$= \frac{|x|}{2n+1}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x|}{2n+1} = 0 < 1$$

That is, the Ratio Test says the series converges absolutely no matter what x might equal, so

the radius of convergence $= R = \infty$

and the interval of convergence $= (-\infty, \infty)$.