11.9 Representation of Functions as Power Series:

mostly using geometric series

<u>Recall</u>: For Geometric series we can say <u>exactly</u> what the series converges to, when it converges.

$$\boxed{a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r} \quad \text{where } |r| < 1,}$$

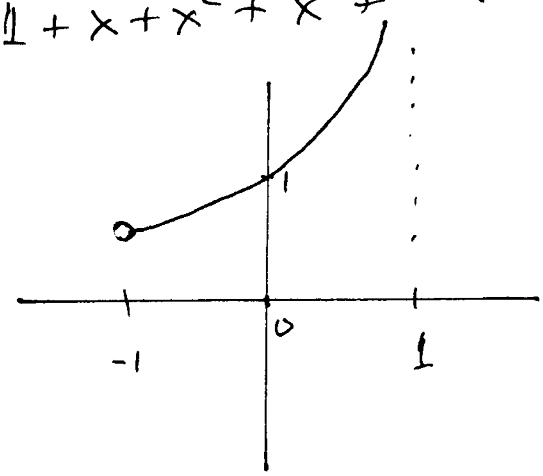
<u>ex</u>: Take $a = 1$, and $r = x$.

$1 + x + x^2 + x^3 + x^4 + \cdots$

Q: For what x does this series converge? when $|x| < 1$ and diverges otherwise.

Q: For those values of x for which the power series converges, what does it converge to?

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} = f(x)$$



interval of convergence $= (-1, 1)$

Radius of convergence $= R = 1$

<u>Remark</u>: For an animated demonstration of the convergence of power series, see TEC on p768 or p777 of the e-book.

ex: Use the formula for the sum of a geometric series to find a power series representation of $f(x) = \frac{1}{1+x^2}$. What is the interval of convergence?

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \frac{a}{1-r}$$ where $a=1$ and $r=-x^2$.

So $$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$
$$= a + ar + ar^2 + ar^3 + \cdots$$
$$= 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + (-x^2)^4 + \cdots$$
$$= 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

The interval of convergence? Where $|r| < 1$.

That is, $|-x^2| < 1$

or $|x^2| < 1$

or $|x| < 1$

ex: Fact: We can integrate power series just like polynomials.

$$\int \frac{1}{1+x^2}\,dx = \int (1 - x^2 + x^4 - x^6 + x^8 - \cdots)\,dx$$

$$\tan^{-1} x = \arctan x = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

What is C? set $x=0$: $\arctan 0 = 0 = C + 0 - 0 + 0 - \cdots \Rightarrow C = 0$

radius of conv $= R = 1$

$$\rightarrow \arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

6) Find a power series representation, and interval of convergence:

$$f(x) = \frac{1}{x+10}$$ Try to put into the form $\frac{a}{1-r} = a \cdot \frac{1}{1-r}$

$$= \frac{1}{10+x} = \frac{1}{10} \cdot \frac{1}{1+\frac{x}{10}} = \frac{1}{10} \cdot \frac{1}{1-\left(\frac{-x}{10}\right)}$$ so $a = \frac{1}{10}$, $r = \frac{-x}{10}$

$$= a + ar + ar^2 + \cdots$$

$$= \frac{1}{10} + \frac{1}{10}\left(\frac{-x}{10}\right) + \frac{1}{10}\left(\frac{-x}{10}\right)^2 + \frac{1}{10}\left(\frac{-x}{10}\right)^3 + \cdots$$

Converges for $|r| < 1$ i.e. $\left|\frac{-x}{10}\right| < 1$ or $|x| < 10$ →

$$= \frac{1}{10} - \frac{x}{10^2} + \frac{x^2}{10^3} - \frac{x^3}{10^4} + \frac{x^4}{10^5} - \cdots$$

7) $$f(x) = \frac{x}{9+x^2} = \frac{x}{9} \cdot \frac{1}{1+\frac{x^2}{9}} = \frac{x}{9} \cdot \frac{1}{1-\left(-\frac{x^2}{9}\right)}$$

so $a = \frac{x}{9}$, $r = \frac{-x^2}{9}$

$$= \frac{x}{9} + \frac{x}{9}\left(\frac{-x^2}{9}\right) + \frac{x}{9} \cdot \left(\frac{-x^2}{9}\right)^2 + \frac{x}{9}\left(\frac{-x^2}{9}\right)^3 + \cdots$$

$$= \frac{x}{9} - \frac{x^3}{9^2} + \frac{x^5}{9^3} - \frac{x^7}{9^4} + \frac{x^9}{9^5} - \cdots$$

interval of convergence? $|r| < 1$ or $\left|\frac{-x^2}{9}\right| < 1$

or $\frac{x^2}{9} < 1$ or $x^2 < 9$

or $\sqrt{x^2} < \sqrt{9}$ or $|x| < 3$

or $-3 < x < 3$.

Another method: Build on known power series

4) $f(x) = \dfrac{5}{1-4x^2}$

Known: $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$ converges for $|x| < 1$

Now, substitute $4x^2$ for x:

$$\frac{1}{1-4x^2} = 1 + 4x^2 + (4x^2)^2 + (4x^2)^3 + \cdots \qquad |4x^2| < 1$$

$$= 1 + 4x^2 + 4^2x^4 + 4^3x^6 + \cdots \qquad |x^2| < \frac{1}{4}$$

So $\sqrt{x^2} < \sqrt{\frac{1}{4}}$ so $|x| < \frac{1}{2}$

Then multiply by 5

$$f(x) = \frac{5}{1-4x^2} = 5 + 5\cdot 4x^2 + 5\cdot 4^2x^4 + 5\cdot 4^3x^6 + \cdots$$

13a) Use differentiation to find a power series for

$f(x) = \dfrac{1}{(1+x)^2}$ What is the radius of convergence?

Know: $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \qquad |x| < 1$

so $\dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = 1 - x + x^2 - x^3 + x^4 \qquad |-x| < 1$

So $|x| < 1$

$$\frac{-1}{(1+x)^2} = \frac{d}{dx}\left[\frac{1}{1+x}\right] = -1 + 2x - 3x^2 + 4x^3 - 5x^4 + \cdots \qquad R = 1$$

so $f(x) = \dfrac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots$

$R = 1$.

Find a power series rep. of this integral.
What is R?

28) $\displaystyle\int \frac{\tan^{-1}x}{x}\,dx$

7th ed

Known: $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$ $\quad R=1$

$$\frac{\tan^{-1}x}{x} = 1 - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + \cdots$$

$$\int \frac{\tan^{-1}x}{x}\,dx = C + x - \frac{x^3}{3^2} + \frac{x^5}{5^2} - \frac{x^7}{7^2} + \cdots$$

$R=1$

30) $\displaystyle\int_0^{0.4} \ln(1+x^4)\,dx$

Known: $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n-1}x^n}{n} + \cdots$ $\quad |x|<1$

Sub x^4 for x: $\ln(1+x^4) = x^4 - \frac{x^8}{2} + \frac{x^{12}}{3} - \frac{x^{16}}{4} + \cdots + \frac{(-1)^{n-1}x^{4n}}{n} + \cdots$ $\quad |x^4|<1$ So $|x|<1$.

$$\int_0^{0.4} \ln(1+x^4)\,dx = \left[\frac{x^5}{5} - \frac{x^9}{2\cdot 9} + \frac{x^{13}}{3\cdot 13} - \frac{x^{17}}{4\cdot 17} + \cdots + \frac{(-1)^{n-1}x^{4n+1}}{n\cdot(4n+1)} + \cdots \right]_0^{0.4}$$

$$= \frac{(0.4)^5}{5} - \frac{(0.4)^9}{18} + \frac{(0.4)^{13}}{39} - \frac{(0.4)^{17}}{68} + \cdots + \frac{(-1)^{n-1}(0.4)^{4n+1}}{n(4n+1)} + \cdots$$

To get six decimal places use the A.S.T. estimation theorem

$= .002048 - .000014563 + .000\,000\,172 - .000\,000\,003 + \cdots$

$= .002\,033\,606 \approx \boxed{.002\,034}$ to six decimal places

In fact, it would suffice to add only the first two terms to get six decimal places.

MORE EXAMPLES (added after class meeting)

ex: Find a power series in x for $f(x) = \dfrac{x^2+1}{x-1}$ and find its radius of convergence.

Solution: First do long division:

$$\begin{array}{r} x + 1 + \frac{2}{x-1} \\ x-1 \,\big)\, x^2 + 0x + 1 \\ x^2 - x \\ \hline x + 1 \\ x - 1 \\ \hline 2 \end{array}$$

OR synthetic division:

$$\begin{array}{c|ccc} 1 & 1 & 0 & 1 \\ & & 1 & 1 \\ \hline & 1 & 1 & 2 \end{array}$$

Either way $\dfrac{x^2+1}{x-1} = x+1 + \dfrac{2}{x-1} = x+1 - \dfrac{2}{1-x}$

Now $\dfrac{-2}{1-x}$ can be written as a geometric series with $a=-2$ and $r = x$; this will converge exactly if $|x|<1$.

$$\frac{-2}{1-x} = -2 - 2x - 2x^2 - 2x^3 - \cdots = -2\sum_{n=0}^{\infty} x^n \quad \text{so that}$$

$$\frac{x^2+1}{x-1} = 1+x - \frac{2}{1-x} = (1-2) + (x-2x) - 2x^2 - 2x^3 - \cdots = \quad \text{with } R=1$$

$$= -1 - x - 2x^2 - 2x^3 - \cdots = -1 - x - 2\sum_{n=2}^{\infty} x^n .$$

ex: Find a power series centered at $x=1$ for $f(x) = \dfrac{1}{x}$. Find its radius of convergence.

$\dfrac{1}{x} = \dfrac{1}{1+(x-1)} = \dfrac{1}{1-[-(x-1)]}$. This can be written as a geometric series with $a=1$ and $r = -(x-1)$, hence will converge if $|-(x-1)| = |x-1| < 1$, i.e. if $-1 < x-1 < 1$ or $0 < x < 2$:

$$\frac{1}{x} = \frac{1}{1-[-(x-1)]} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

We have been using this theorem:

Theorem: If $\sum c_n(x-a)^n$ has a radius of convergence $R>0$, then the function defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and continuous) on the interval $(a-R, a+R)$ and

(i) $$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

(ii) $$\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

Moreover, the radius of convergence for the series are both R.

7th ed. #18) Find a power series representation for $f(x) = \left(\frac{x}{2-x}\right)^3$ and find its radius of convergence.

Begin with $\frac{1}{2-x} = \frac{1}{2}\cdot\frac{1}{1-\left(\frac{x}{2}\right)}$, the sum of a geometric series with $a=\frac{1}{2}$, $r=\frac{x}{2}$

So $$\frac{1}{2}\cdot\frac{1}{1-\left(\frac{x}{2}\right)} = \frac{1}{2} + \frac{1}{2}\cdot\frac{x}{2} + \frac{1}{2}\cdot\frac{x^2}{2^2} + \cdots = \sum_{n=0}^{\infty}\frac{1}{2}\cdot\frac{x^n}{2^n}, \quad \left|\frac{x}{2}\right|<1 \text{ or } |x|<2.$$

$$\frac{1}{(2-x)^2} = \frac{d}{dx}\left[\frac{1}{2-x}\right] = \frac{d}{dx}\left[\frac{1}{2} + \frac{x}{2^2} + \frac{x^2}{2^3} + \cdots\right]$$
$$= \frac{1}{2^2} + \frac{2x}{2^3} + \frac{3x^2}{2^4} + \frac{4x^3}{2^5} + \cdots = \sum_{n=1}^{\infty}\frac{n x^{n-1}}{2^{n+1}} \quad \text{for } |x|<2.$$

$$= \frac{d}{dx}(2-x)^{-2} = \frac{d}{dx}\left[\frac{1}{(2-x)^2}\right] = \frac{d}{dx}\left[\frac{1}{2^2} + \frac{2x}{2^3} + \frac{3x^2}{2^4} + \cdots\right]$$

$$2(2-x)^{-3} = \frac{2}{(2-x)^3} = \frac{2\cdot 1}{2^3} + \frac{3\cdot 2x}{2^4} + \frac{4\cdot 3x^2}{2^5} + \cdots = \sum_{n=2}^{\infty}\frac{n(n-1)x^{n-2}}{2^{n+1}}$$

[cont'd]

#18 cont'd) Multiply by $\frac{1}{2}$:

$$\frac{1}{(2-x)^3} = \frac{1}{2}\cdot\frac{2}{(2-x)^3} = \frac{1}{2}\sum_{n=2}^{\infty}\frac{n(n-1)x^{n-2}}{2^{n+1}} = \sum_{n=2}^{\infty}\frac{n(n-1)x^{n-2}}{2^{n+2}}$$

$$= \frac{2\cdot 1}{2^4} + \frac{3\cdot 2x}{2^5} + \frac{4\cdot 3x^2}{2^6} + \frac{5\cdot 4x^3}{2^7} + \cdots$$

Now, multiply by x^3:

$$f(x) = \frac{x^3}{(2-x)^3} = \left(\frac{x}{2-x}\right)^3 = x^3\sum_{n=2}^{\infty}\frac{n(n-1)x^{n-2}}{2^{n+2}} = \sum_{n=2}^{\infty}\frac{n(n-1)x^{n+1}}{2^{n+2}}$$

$$= \frac{2\cdot 1}{2^4}x^3 + \frac{3\cdot 2}{2^5}x^4 + \frac{4\cdot 3}{2^6}x^5 + \frac{5\cdot 4}{2^7}x^6 + \cdots$$

which will converge if $|x|<2$.

#34) Show that $y = f(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

satisfies $y'' + y = 0$. Note: We will see that $f(x) = \cos x$, converging for all real numbers.

$$y' = f'(x) = \sum_{n=1}^{\infty}(-1)^n\frac{2n\,x^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty}(-1)^n\frac{x^{2n-1}}{(2n-1)!} = -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots$$

$$y'' = f''(x) = \sum_{n=1}^{\infty}(-1)^n\frac{(2n-1)x^{2n-2}}{(2n-1)!} = \sum_{n=1}^{\infty}(-1)^n\frac{x^{2n-2}}{(2n-2)!} = -1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots$$

$$y'' + y = \sum_{n=1}^{\infty}(-1)^n\frac{x^{2n-2}}{(2n-2)!} + \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} = \sum_{k=0}^{\infty}(-1)^{k+1}\frac{x^{2k}}{(2k)!} + \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}$$

Let $n = k+1$ so that $2n = 2k+2$, $2n-2 = 2k$; if $n=1$ then $k=0$

Let $k=n$

$$= -\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!} + \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}$$

$$= 0$$