# 11.10 Taylor and Maclaurin Series

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

Q1: If a function can be represented as a power series what can we say about the $c_n$'s?

Q2: Which functions can be represented by power series?

A1: Observe: $f(a) = c_0 + 0 + 0 + 0 + \cdots = \boxed{c_0}$

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$$

$$f'(a) = c_1 + 2c_2\cancel{(a-a)}^{0} + 3c_3\cancel{(a-a)}^{0\,2} + \cdots = \boxed{c_1}$$

$$f''(x) = 2c_2 + 3\cdot 2c_3(x-a) + 4\cdot 3c_4(x-a)^2 + \cdots$$

$$f''(a) = 2c_2 + 3\cdot 2c_3(a-a) + 4\cdot 3c_4(a-a)^2 + \cdots = \boxed{2c_2}$$

$$f'''(x) = 3\cdot 2c_3 + 4\cdot 3\cdot 2c_4(x-a) + 5\cdot 4\cdot 3(x-a)^2 + \cdots$$

$$f'''(a) = 3\cdot 2\cdot 1\cdot c_3 + 4\cdot 3\cdot 2c_4\cancel{(a-a)}^{0} + 5\cdot 4\cdot 3\cancel{(x-a)}^{0\,2} + \cdots = \boxed{3!\,c_3}$$

$\vdots$

$$f^{(n)}(a) = n!\,c_n + 0 + 0 + 0 = \boxed{n!\,c_n}$$

$$\boxed{c_n = \frac{f^{(n)}(a)}{n!}}$$

That is, to find $c_n$

1. Take $n^{th}$ derivative of $f(x)$ to get $f^{(n)}(x)$
2. Plug in $a$ for $x$: $f^{(n)}(a)$
3. Divide by $n!$: $\frac{f^{(n)}(a)}{n!} = c_n$

ex: For $f(x) = \cos x$, and $a=0$, find the power series that is (our only candidate for a) representation cos x.

| $n$ | $f^{(n)}(x)$ | $f^{(n)}(0)$ | $f^{(n)}(0)/n! = c_n$ |
|---|---|---|---|
| 0 | $\cos x$ | $\cos 0 = 1$ | $1/0! = 1/1 = 1 = c_0$ |
| 1 | $-\sin x$ | $-\sin 0 = 0$ | $0/1! = 0 = c_1$ |
| 2 | $-\cos x$ | $-\cos 0 = -1$ | $-1/2! = -1/2 = c_2$ |
| 3 | $+\sin x$ | $\sin 0 = 0$ | $0/3! = 0 = c_3$ |
| 4 | $\cos x$ | $\cos 0 = 1$ | $1/4! = \frac{1}{24} = c_4$ |

Aha! I see the pattern for $c_0, c_1, c_2 \ldots$

we hope

So $\cos x \overset{\downarrow}{=} 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \ldots$

This is the Maclaurin series for $f(x) = \cos x$.

Pop quiz: Find the Taylor series at $a=1$ for $f(x)=e^x$

| $n$ | $f^{(n)}(x)$ | $f^{(n)}(1)$ | $f^{(n)}(1)/n! = c_n$ |
|---|---|---|---|
| 0 | $e^x$ | $e^1 = e$ | $e/0! = e$ |
| 1 | $e^x$ | $e$ | $e/1! = e$ |
| 2 | $e^x$ | $e$ | $e/2! = e/2$ |
| 3 | $e^x$ | $e$ | $e/3! = e/6$ |
| 4 | $e^x$ | $e$ | $e/4! = e/24$ |

$$e^x \overset{\text{we hope}}{=} e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{6}(x-1)^3 + \cdots$$

$$\simeq 2.718 + 2.718(x-1) + 1.36(x-1)^2 + .453(x-1)^3 + .113(x-1)^4 + \cdots$$

Terminology: We just found the Taylor series, centered at 1, for $f(x)=e^x$.

$T_n(x) = n^{th}$ Taylor polynomial

$T_0(x) = e$

$T_1(x) = e + e(x-1)$

$T_2(x) = e + e(x-1) + \frac{e}{2}(x-1)^2$

$T_3(x) = e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{6}(x-1)^3$

$$f(x) = e^x = \underbrace{T_n(x)}_{n^{th}\text{ Taylor polynomial}} + \underbrace{R_n(x)}_{n^{th}\text{ Taylor remainder}}$$

We want to get a handle on $R_n(x)$.

Not in textbook: Use of Tabular Integration by parts

[See www.stewartcalculus.com]

Begin with

$$\int_a^x f'(t)\,dt = f(x) - f(a)$$

by the Fundamental Thm. of Calculus

Remark: Treat $x$ and $a$ as constants and $t$ as the variable.

| Signs | Diff | Int |
|---|---|---|
| $+ \longrightarrow$ | $f'(t)$ | $1$ |
| $- \longrightarrow$ | $f''(t)$ | $t-x$ |
| $+ \longrightarrow$ | $f'''(t)$ | $\frac{1}{2}(t-x)^2$ |

check: $\frac{d}{dt}[t-x] = 1-0 = 0$

This says:

$$\int_a^x f'(t)\,dt = f'(t)(t-x)\Big|_a^x - \frac{f''(t)}{2!}(t-x)^2\Big|_a^x + \int_a^x \frac{f'''(t)}{2!}(t-x)^2\,dt$$

$$= f'(x)\cancel{(x-x)}^{0} - f'(a)(a-x)$$

$$- \frac{f''(x)}{2!}\cancel{(x-x)}^{0} + \frac{f''(a)}{2!}(a-x)^2$$

$$+ \int_a^x \frac{f'''(t)}{2!}(t-x)^2\,dt$$

$$= -f'(a)(a-x) + \frac{f''(a)}{2!}(a-x)^2 + R_2(x)$$

$$f(x) - f(a) = f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + R_2(x)$$

$$f(x) = \underbrace{f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2}_{T_2(x)} + R_2(x)$$

(To be continued)

Added after class:

We have shown that if $f'''(t)$ exists (and is continuous) near $a$,

$$f(x) = T_2(x) + R_2(x)$$

where

$$T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

= the $2^{nd}$-degree Taylor polynomial of $f$ at $a$

and

$$R_2(x) = \int_a^x \frac{f'''(t)}{2!}(t-x)^2\,dt = \int_a^x \frac{f'''(t)}{2!}(x-t)^2\,dt$$

= the "integral form" of the remainder term of the Taylor series

For another form for $R_2(x)$, one can show there is a number $z$ between $x$ and $a$ such that $\int_a^x \frac{f'''(t)}{2!}(t-x)^2\,dt = \int_a^x \frac{f'''(z)}{2!}(t-x)^2\,dt$

$$= \frac{f'''(z)}{2!}\left.\frac{(t-x)^3}{3}\right|_a^x = \frac{f'''(z)}{3!}\overbrace{(x-x)^3}^{0} - \frac{f'''(z)}{3!}(a-x)^3$$

$$= \frac{f'''(z)}{3!}(x-a)^3$$ = the "Lagrange form" of the remainder term

[Remark: $f'''(z)$ is a sort of "weighted average" of $f'''(t)$.]

More generally we have:

Theorem: If $f^{(n+1)}$ is continuous on an open interval $I$ that contains $a$, and $x$ is in $I$, then there exists a number $z$ between $a$ and $x$ such that

$$\boxed{R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}}$$

For this $R_n(x)$ we can write

$$\boxed{f(x) = T_n(x) + R_n(x)}$$ where

$$\boxed{T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

= the $n^{th}$-degree Taylor polynomial of $f$ at $a$.