11.10 Taylor Series (cont'd)

Problems using the Power Series handout

7th ed) 32) Find the Maclaurin Series for $f(x) = e^x + 2e^{-x}$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$ Sub $-x$ for x:

$$e^{-x} = 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-3)^3}{3!} + \cdots$$

$$= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

$$2e^{-x} = 2 - 2x + \frac{2x^2}{2!} - \frac{2x^3}{3!} + \cdots$$

$$f(x) = e^x + 2e^{-x} = 3 - x + \frac{3x^2}{2!} - \frac{x^3}{3!} + \frac{3x^4}{4!} - \frac{x^5}{5!} + \cdots$$

33) $f(x) = x\cos\left(\frac{1}{2}x^2\right)$

Begin with $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

Sub $\frac{x^2}{2}$ for x:

$$\cos\left(\frac{1}{2}x^2\right) = 1 - \frac{1}{2!}\cdot\left(\frac{x^2}{2}\right)^2 + \frac{1}{4!}\left(\frac{x^2}{2}\right)^4 - \frac{1}{6!}\left(\frac{x^2}{2}\right)^6 + \cdots$$

$$= 1 - \frac{1}{2!}\cdot\frac{x^4}{2^2} + \frac{1}{4!}\cdot\frac{x^8}{2^4} - \frac{1}{6!}\cdot\frac{x^{12}}{2^6} + \cdots$$

$$f(x) = x\cos\left(\frac{1}{2}x^2\right) = x - \frac{1}{2!}\cdot\frac{x^5}{2^2} + \frac{1}{4!}\cdot\frac{x^9}{2^4} - \frac{1}{6!}\cdot\frac{x^{13}}{2^6} + \cdots$$

Euler's formula

$$\boxed{e^{ix} = \cos x + i \sin x}$$ where $i = \sqrt{-1}$

Reason:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

so $$i \sin x = ix - \frac{ix^3}{3!} + \frac{ix^5}{5!} - \cdots$$

$$\cos x + i \sin x = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \cdots$$

match $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

Sub ix for x, and use $i^1 = 1$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \cdots$$

match $\rightarrow$ $$= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} + \cdots$$

Remark: Let $x = \pi$

$$e^{i\pi} = \cos \pi + i \sin \pi = -1 + 0i = -1$$

$$\boxed{e^{i\pi} + 1 = 0}$$ ← Your next tattoo.

48) Evaluate as an infinite series

$$\int \frac{e^x - 1}{x}\,dx$$

Begin with: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

$$e^x - 1 = x\left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots\right)$$

$$\frac{e^x - 1}{x} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots$$

$$\int \frac{e^x - 1}{x}\,dx = C + x + \frac{x^2}{2\cdot 2!} + \frac{x^3}{3\cdot 3!} + \frac{x^4}{4\cdot 4!} + \cdots$$

57) Use series to evaluate

$$\lim_{x \to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5}$$

Begin with: $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$$-x + \frac{x^3}{6} \qquad -x + \frac{x^3}{3!}$$

$$\sin x - x + \frac{x^3}{6} = \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

$$\frac{\sin x - x + \frac{x^3}{6}}{x^5} = \frac{1}{5!} - \frac{x^2}{7!} + \frac{x^4}{9!} - \cdots$$

$$\lim_{x \to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5} = \frac{1}{5!} = \frac{1}{5\cdot 4\cdot 3\cdot 2\cdot 1} = \frac{1}{120}$$

Q2: [from yesterday]

We have a function $f(x) = \cos x$ and we have its Maclaurin series $p(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ which converges on $(-\infty, \infty)$.

How do we know $f(x)$ and $p(x)$ are the same function.

Answer: The remainder formula $R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$

for z between x and a, will help answer the question:

Know: $\cos x = T_3(x) + R_3(x)$

$$T_3(x) = 1 + 0x - \frac{1}{2!}x^2 + 0x^3$$

$$R_3(x) = \frac{f^{(4)}(z)}{4!}(x-0)^4$$

Are these similar functions?

$$\left|\cos x - T_3(x)\right| = \left|R_3(x)\right|$$

$$= \left|\frac{f^{(4)}(z)}{4!}x^4\right| = \frac{\left|f^{(4)}(z)\right|}{24}|x|^4$$

$$= \frac{|\cos z|}{24}|x|^4 \le \frac{1}{24}|x|^4$$

more generally:

$$\left|\cos x - T_n(x)\right| \le \frac{1}{n!}|x|^n \to 0 \text{ as } n \to \infty \text{ for all } x$$

This says $\lim_{n\to\infty} T_n(x) = p(x) = \cos x$.

Binomial series

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

has radius of convergence $= R = 1$

ex: write out the Maclaurin series for

$$f(x) = \sqrt{1+x} = (1+x)^{1/2} \qquad \text{so } k = 1/2$$

$$= 1 + \frac{1}{2}x + \frac{1}{2}\left(\frac{-1}{2}\right)\frac{x^2}{2!} + \frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\frac{x^3}{3!}$$

$$+ \frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)\frac{x^4}{4!} + \cdots$$

ex: Use $T_2(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2$ to approximate $\sqrt{1.2}$.

$$\sqrt{1.2} = f(0.2) = \sqrt{1+0.2}$$

$$\approx 1 + \frac{1}{2}(0.2) - \frac{1}{8}(0.2)^2$$

$$= 1 + 0.1 - \frac{.4}{8}$$

$$= 1 + .1 - .05 = 1.095$$

Actual value: $\sqrt{1.2} = 1.09544511 5$

Actual remainder $= f(0.2) - T_2(0.2) = 1.09544511 5 - 1.095 = 4.45 \cdot 10^{-4}$

Predicted remainder $= |R_2(0.2)| = \left|\frac{f'''(z)}{3!}(0.2)^3\right| \leq \frac{3}{2^3}\frac{(0.2)^3}{3!} = 5 \cdot 10^{-4}$

where we're using that $f'''(z) = \frac{3}{2^3}(1+z)^{-5/2} \leq \frac{3}{2^3}$ whenever $0 < z < 1$.

Added after class: Derivation of the Maclaurin series for $f(x) = (1+x)^k$ where k is any real number

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$f^{(n)}(0)/n! = c_n$
0	$(1+x)^k$	1	1
1	$k(1+x)^{k-1}$	k	k
2	$k(k-1)(1+x)^{k-2}$	$k(k-1)$	$\frac{k(k-1)}{2!}$
3	$k(k-1)(k-2)(1+x)^{k-3}$	$k(k-1)(k-2)$	$\frac{k(k-1)(k-2)}{3!}$
4	$k(k-1)(k-2)(k-3)(1+x)^{k-4}$	$k(k-1)(k-2)(k-3)$	$\frac{k(k-1)(k-2)}{4!}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

$$\therefore \quad (1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots + \frac{k(k-1)\cdots(k-n+1)}{n!}x^n + \cdots$$

Remark: It can be shown that

if $k \geq 0$, the interval of convergence is $[-1, 1]$

if $-1 < k \leq 0$, " " " " is $(-1, 1]$

if $k \leq -1$, " " " " is $(-1, 1)$

ex: Find the Maclaurin Series for $f(x) = \frac{1}{\sqrt{1+x}} = (1+x)^{-1/2}$, so $k = -1/2$

$$1 + \left(-\tfrac{1}{2}\right)x + \left(\tfrac{-1}{2}\right)\left(\tfrac{-3}{2}\right)\frac{x^2}{2!} + \left(\tfrac{-1}{2}\right)\left(\tfrac{-3}{2}\right)\left(\tfrac{-5}{2}\right)\frac{x^3}{3!} + \left(\tfrac{-1}{2}\right)\left(\tfrac{-3}{2}\right)\left(\tfrac{-5}{2}\right)\left(\tfrac{-7}{2}\right)\frac{x^4}{4!} + \cdots$$

$$= 1 - \frac{1}{2}x + \frac{1\cdot 3}{2^2}\cdot\frac{x^2}{2!} - \frac{1\cdot 3\cdot 5}{2^3}\cdot\frac{x^3}{3!} + \frac{1\cdot 3\cdot 5\cdot 7}{2^4}\cdot\frac{x^4}{4!} + \cdots$$

ex: Find the Maclaurin series for $g(x) = \frac{1}{\sqrt{1-x^2}}$.

Method: Substitute $-x^2$ for x in the previous example:

$$1 - \frac{1}{2}(-x^2) + \frac{1\cdot 3}{2^2}\cdot\frac{(-x^2)^2}{2!} - \frac{1\cdot 3\cdot 5}{2^3}\cdot\frac{(-x^2)^3}{3!} + \frac{1\cdot 3\cdot 5\cdot 7}{2^4}\cdot\frac{(-x^2)^4}{4!} + \cdots$$

$$= 1 + \frac{1}{2}x^2 + \frac{1\cdot 3}{2^2}\cdot\frac{x^4}{2!} + \frac{1\cdot 3\cdot 5}{2^3}\cdot\frac{x^6}{3!} + \frac{1\cdot 3\cdot 5\cdot 7}{2^4}\cdot\frac{x^8}{4!} + \cdots$$

ex: Find the Maclaurin series for $\sin^{-1} x = \int \frac{1}{\sqrt{1-x^2}}\,dx$

Method: Integrate, term by term, the series from the previous example:

$$\sin^{-1} x = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot 3}{2^2}\cdot\frac{x^5}{5\cdot 2!} + \frac{1\cdot 3\cdot 5}{2^3}\cdot\frac{x^7}{7\cdot 3!} + \frac{1\cdot 3\cdot 5\cdot 7}{2^4}\cdot\frac{x^9}{9\cdot 4!} + \cdots$$

$$= x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^7}{7} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{x^9}{9} + \cdots$$