11.10 Even More on Taylor Series and 11.11

7th ed #66) Use the list of common power series to find the sum

$$\sum_{n=0}^{\infty} \frac{3^n}{5^n n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \left(\frac{3}{5}\right)^n$$

Since $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot x^n$ if we let $x = 3/5$

$$\boxed{e^{3/5}} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{3}{5}\right)^n$$

67) $$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1}(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\pi}{4}\right)^{2n+1}$$

Since $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ If we let $x = \frac{\pi}{4}$

$$\boxed{\frac{\sqrt{2}}{2}} = \sin\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\pi}{4}\right)^{2n+1}$$

ex: Goal: Find $\cos 61^\circ$ to whatever precision we want.

part a) Since $61^\circ = 61 \cdot \frac{\pi}{180} = \frac{61\pi}{180}$ is close to $\frac{60\pi}{180} = \frac{\pi}{3}$

let's find the Taylor series at $x = \frac{\pi}{3}$ of $\cos x = f(x)$

Method 1 make a table

n	$f^{(n)}(x)$	$f^{(n)}(\frac{\pi}{3})$	$f^{(n)}(\pi/3)/n! = c_n$
0	$\cos x$	$\frac{1}{2}$	$\frac{1}{2}$
1	$-\sin x$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$
2	$-\cos x$	$-1/2$	$-\frac{1}{2} \cdot \frac{1}{2!}$
3	$\sin x$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2} \cdot \frac{1}{3!}$
4	$\cos x$	$1/2$	$\frac{1}{2} \cdot \frac{1}{4!}$
			$\vdots$

Taylor series:

$$\frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{2}\frac{(x-\pi/3)^2}{2!} + \frac{\sqrt{3}}{2}\frac{(x-\pi/3)^3}{3!} + \frac{1}{2}\frac{(x-\pi/3)^4}{4!} + \cdots$$

Method 2: Using the List of Power Series.

$$f(x) = \cos x = \cos\left[\left(x-\tfrac{\pi}{3}\right)+\tfrac{\pi}{3}\right]$$

$$= \cos\left(x-\tfrac{\pi}{3}\right)\cos\tfrac{\pi}{3} - \sin\left(x-\tfrac{\pi}{3}\right)\sin\tfrac{\pi}{3}$$

$$= \frac{1}{2}\cos\left(x-\tfrac{\pi}{3}\right) - \frac{\sqrt{3}}{2}\sin\left(x-\tfrac{\pi}{3}\right)$$

Now:
(purple sheet)

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

☆ $$\frac{1}{2}\cos x = \frac{1}{2} - \frac{1}{2}\cdot\frac{x^2}{2!} + \frac{1}{2}\cdot\frac{x^4}{4!} - \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

☆ $$-\frac{\sqrt{3}}{2}\sin x = -\frac{\sqrt{3}}{2}x + \frac{\sqrt{3}}{2}\cdot\frac{x^3}{3!} - \frac{\sqrt{3}}{2}\frac{x^5}{5!}$$

Sub $\left(x-\frac{\pi}{3}\right)$ for x:
and add:

$$f(x) = \frac{1}{2}\cos\left(x-\tfrac{\pi}{3}\right) - \frac{\sqrt{3}}{2}\sin\left(x-\tfrac{\pi}{3}\right)$$

$$= \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x-\tfrac{\pi}{3}\right) - \frac{1}{2}\frac{\left(x-\pi/3\right)^2}{2!} + \frac{\sqrt{3}}{2}\frac{\left(x-\pi/3\right)^3}{3!} + \frac{1}{2}\frac{\left(x-\pi/3\right)^4}{4!}$$

part b) $$f(x) \approx T_2(x) = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x-\tfrac{\pi}{3}\right) - \frac{1}{2}\frac{\left(x-\pi/3\right)^2}{2!}$$

Use this $T_2(x)$ to approximate $\cos 61°$

$$T_2\left(\frac{61\pi}{180}\right) = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(\frac{61\pi}{180} - \frac{\pi}{3}\right) - \frac{1}{2\cdot 2!}\left(\frac{61\pi}{180} - \frac{\pi}{3}\right)^2$$

That is,

$$T_2\left(\frac{61\pi}{180}\right) = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(\frac{\pi}{180}\right) - \frac{1}{4}\left(\frac{\pi}{180}\right)^2$$

$$= .5 - .015\ 114\ 994702 - .000\ 076\ 154\ 555$$

$$= 0.484\ \cancel{732\ 6966}\ \ 808\ 850\ 943$$

By calculator, $\cos 61^\circ = .484\ 809\ 620246$

part c) Use Taylor's inequality: $\left|R_n(x)\right| \le \frac{M}{(n+1)!}\left|x-a\right|^{n+1}$ for $|x-a|<d$
to find a bound on the error of the approximation found in part b

where M is a number such that

$$\left|f^{(n+1)}(x)\right| \le M \quad \text{for } |x-a|<d .$$

what's behind this: Lagrange form of remainder term $= R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)$

In our example, since $f(x) = \cos x$ and $f'''(x) = \sin x$

we can take $M = 1$, because $|\sin x| \le 1$ for any value of x.

So $$\left|R_2(x)\right| \le \frac{1}{3!}\left(x - \frac{\pi}{3}\right)^3$$

In particular, if $x = 61^\circ = \frac{61\pi}{180}$ so $x - \frac{\pi}{3} = \frac{\pi}{180}$

then $$\left|R_2\left(\frac{61\pi}{180}\right)\right| \le \frac{1}{3!}\left(\frac{\pi}{180}\right)^3 = \frac{1}{6}\left(\frac{\pi}{180}\right)^3$$ = an upper bound for the remainder

$$= 8.86 \cdot 10^{-7}$$

$$= .000\ 000\ 886$$

The Actual remainder $= \left|\cos 61^\circ - T_2\left(\frac{61\pi}{180}\right)\right| = .484\ 809\ 620\ 246 - .484\ 808\ 850\ 943$

$$= R_2\left(\frac{61\pi}{180}\right) = 7.69 \cdot 10^{-7} = .000\ 000\ 769\ 303$$

Added after class: Multiplication and Division of Power Series

Remark: Power series may be multiplied and divided like polynomials.

example: Find the first few terms of the Maclaurin series for $f(x) = \sec x = \frac{1}{\cos x}$ by dividing 1 by the Maclaurin series for $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$$
\begin{array}{r|l}
 & 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \cdots \\
\hline
1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots & 1 + 0x^2 + 0x^4 + 0x^6 + \cdots \\
 & 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \cdots \\
\cline{2-2}
 & \frac{1}{2}x^2 - \frac{1}{24}x^4 + \frac{1}{720}x^6 - \cdots \\
 & \frac{1}{2}x^2 - \frac{1}{4}x^4 + \frac{1}{48}x^6 - \cdots \\
\cline{2-2}
 & \frac{5}{24}x^4 - \frac{7}{360}x^6 + \cdots \\
 & \frac{5}{24}x^4 - \frac{5}{48}x^6 + \cdots \\
\cline{2-2}
 & \frac{61}{720}x^6 - \cdots \\
 & \frac{61}{720}x^6 - \cdots \\
\cline{2-2}
\end{array}
$$

So the Maclaurin series for $f(x) = \sec x$ (showing the first four nonzero terms) is:

$$\sec x = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \cdots$$