10.1 Parametric Curves (cont'd)

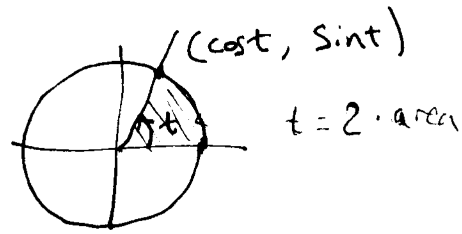
Some standard parametrizations

ex: unit circle

$x = \cos t$

$y = \sin t$

$0 \le t \le 2\pi$

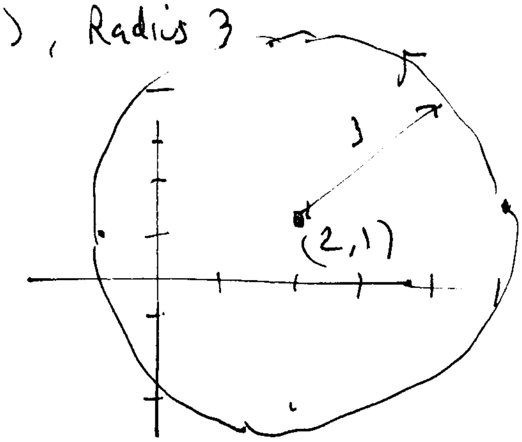


ex: circle, centered at $(2,1)$, Radius 3

$x = 2 + 3\cos t$

$y = 1 + 3\sin t$

$0 \le t \le 2\pi$



eliminate the parameter:

$x - 2 = 3\cos t \implies (x-2)^2 = 9\cos^2 t$

$y - 1 = 3\sin t \qquad (y-1)^2 = 9\sin^2 t$

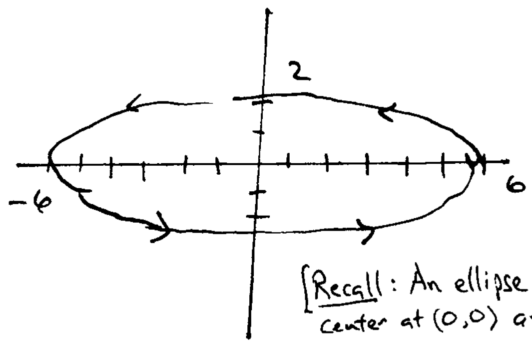
$(x-2)^2 + (y-1)^2 = 9(\cos^2 t + \sin^2 t)$

circle
center $= (2,1)$
radius $= 3$
$\longrightarrow$ $\boxed{(x-2)^2 + (y-1)^2 = 9}$

This Lacks any way to orient the curve, or to trim the curve.

e.g. $0 \le t \le \pi/2$ would model a quarter circle.

ex: $x = 6\cos t$
$y = 2\sin t$
$0 \le t \le 2\pi$



[Recall: An ellipse with center at (0,0) and vertices at ($\pm a$,0), major axis=2a, minor axis = 2b, has equation
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.]

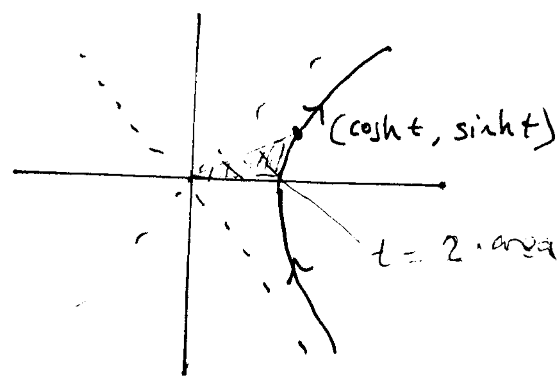
Eliminate the parameter:

$\frac{x}{6} = \cos t \quad \Rightarrow \quad \frac{x^2}{36} = \cos^2 t$

$\frac{y}{2} = \sin t \qquad\quad \frac{y^2}{4} = \sin^2 t$

$\frac{x^2}{36} + \frac{y^2}{4} = 1$ ← ellipse $a=6$, $b=2$

ex: $x = \cosh t = \frac{e^t + e^{-t}}{2}$
$y = \sinh t = \frac{e^t - e^{-t}}{2}$
$-\infty < t < \infty$



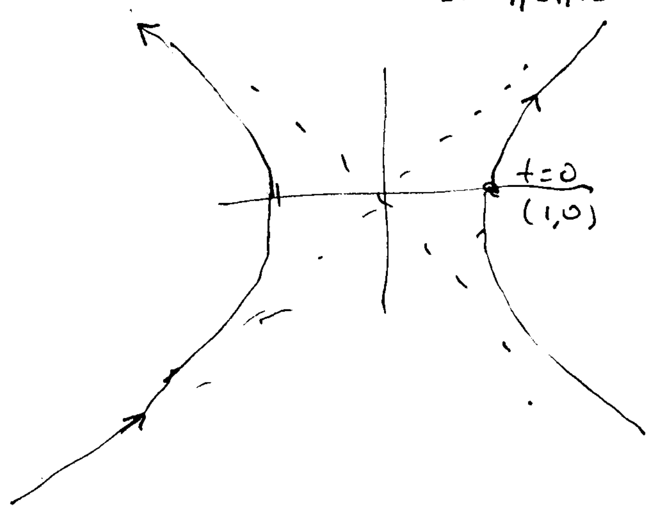
$x^2 = \cosh^2 t$
$y^2 = \sinh^2 t$

$x^2 - y^2 = \cosh^2 t - \sinh^2 t$

$x^2 - y^2 = 1$ [A "circular hyperbola" is one in which $a=b$
so $\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$, just as a "circular ellipse"
(i.e. a circle) is one in which $a = b$
so $\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$ or $x^2+y^2=a^2$.]

ex: $\begin{cases} x = \sec t \\ y = \tan t \end{cases}$

$0 \le t \le 2\pi$



$x^2 = \sec^2 t$
$y^2 = \tan^2 t$

$x^2 - y^2 = \sec^2 t - \tan^2 t$
$x^2 - y^2 = 1$ ← hyperbola $a=1$, $b=1$

[Recall: A hyperbola centered at (0,0) with vertices $(\pm a, 0)$, transverse axis $= 2a$, conjugate axis $= 2b$, has equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$]

Remark: (1) If $y = f(x)$ we can make parametric equations out of this: $\begin{cases} x = t \\ y = f(t) \end{cases}$

(2) If $x = g(y)$ let $\begin{cases} x = g(t) \\ y = t \end{cases}$

ex we want to graph $x = 4 - y^2$, but only the portion in the right half plane, $x \ge 0$.

Take: $\begin{cases} x = 4 - t^2 \\ y = t \end{cases}$ $-2 \le t \le 2$

ex: we want orientation reversed? No prob.

$$\begin{cases} x = 4 - (-t)^2 = 4 - t^2 \\ y = (-t) \quad = -t \end{cases}$$

$-2 \le -t \le 2 \iff -2 \le t \le 2$

10.2 Calculus of Parametric Curves
[Skip area and surface area]

Tangents

If $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$ are parametric equations

then $\boxed{\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}}$ [so we can find slope of a tangent line without eliminating the parameter.]

ex: $x = 2 + 3\cos t$
$y = 1 + 3\sin t$

a) What is $\dfrac{dy}{dx}$ when $t = \dfrac{3\pi}{4}$?

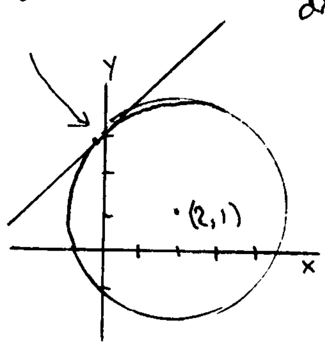
$$\frac{dx}{dt} = -3\sin t$$

$$\frac{dy}{dt} = 3\cos t$$

$$\frac{dy/dt}{dx/dt} = \frac{dy}{dx} = \frac{3\cos t}{-3\sin t} = -\cot t$$

So if $t = \frac{3\pi}{4}$, $\left.\dfrac{dy}{dt}\right|_{t=3\pi/4} = -\cot\frac{3\pi}{4} = -(-1) = \boxed{1}$

point $= (-0.12, 3.12)$
slope $= 1$



b) What is the equation of the tangent line? when $t = 3\pi/4$

$x = 2 + 3\cos\frac{3\pi}{4}$
$= \boxed{2 + 3\left(-\frac{\sqrt{2}}{2}\right)} \approx \boxed{-0.12}$

$y = 1 + 3\sin\frac{3\pi}{4}$
$= \boxed{1 + 3\left(\frac{\sqrt{2}}{2}\right)} \approx \boxed{3.12}$

Line: [using pt-slope form; then slope-intercept form.]

$$y - \left(1 + \frac{3\sqrt{2}}{2}\right) = 1 \cdot \left[x - \left(2 - \frac{3\sqrt{2}}{2}\right)\right]$$

$$y - 1 - \frac{3\sqrt{2}}{2} = x - 2 + \frac{3\sqrt{2}}{2}$$

$$\boxed{y = x - 1 + 3\sqrt{2}}$$

or approximately $y - 3.12 = x + 0.12$ or $y = x + 3.24$

2nd derivative?

Since $\dfrac{dy}{dx} = \dfrac{\frac{d}{dt}[y]}{\frac{d}{dt}[x]}$ then if $y' = \dfrac{dy}{dx}$

$$\frac{d[y']}{dx} = \frac{\frac{d}{dt}[y']}{\frac{d}{dt}[x]}$$ That is

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}$$

ex: Find $\dfrac{d^2y}{dx^2}$ in the previous example:

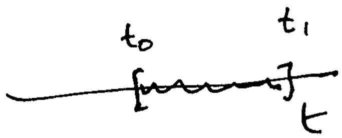
$\dfrac{dx}{dt} = -3\sin t$ and $\dfrac{dy}{dx} = -\cot t$

$\dfrac{dy}{dt} = 3\cos t$

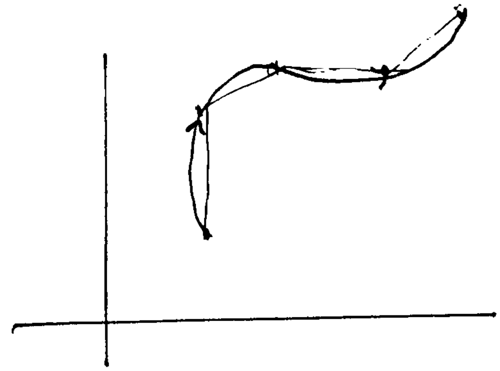
so $$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}} = \frac{\frac{d}{dt}[-\cot t]}{-3\sin t} = \frac{\csc^2 t}{-3\sin t}$$
$$= -\frac{1}{3}\csc^3 t$$

Remark [added after class]: This result for $\frac{d^2y}{dx^2}$ confirms that the parametric curve (i.e. the circle) is concave down when $\frac{d^2y}{dx^2} = -\frac{1}{3}\csc^3 t$ is negative, namely when $0 < t < \pi$ so that t is in quadrants I or II, where $\sin t$ (hence $\csc t$) is positive.

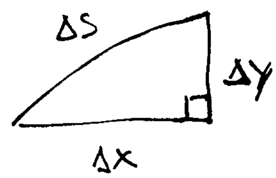
Arclength



$x = f(t)$
$y = f(t)$



zoom in:
one line segment



$$\Delta s^2 = \Delta x^2 + \Delta y^2$$

$$\Delta s = \sqrt{\Delta x^2 + \Delta y^2}$$

Total length $\approx \sum \Delta s = \sum \sqrt{\Delta x^2 + \Delta y^2}$

$$= \sum \left(\sqrt{\frac{\Delta x^2}{\Delta t^2} + \frac{\Delta y^2}{\Delta t^2}} \right) \Delta t$$

Take limits:
$$\boxed{\text{Total Arc Length} = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

ex: Find the length of the circle

$$\begin{cases} x = 2 + 3\cos t \\ y = 1 + 3\sin t \end{cases}$$

$$0 \le t \le 2\pi$$

$$\frac{dx}{dt} = -3\sin t$$

$$\frac{dy}{dt} = 3\cos t$$

$$\left(\frac{dx}{dt}\right)^2 = 9\sin^2 t$$

$$\left(\frac{dy}{dt}\right)^2 = 9\cos^2 t$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9(\sin^2 t + \cos^2 t) = 9$$

$$\text{Speed} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{9} = 3$$

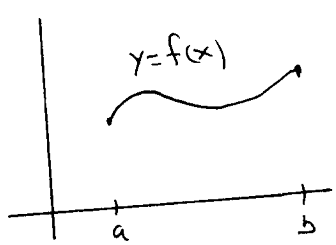
$$\text{arc length} = \int_{t=0}^{t=2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} 3\, dt$$

$$= 3t\Big]_0^{2\pi} = 3[2\pi - 0] = \boxed{6\pi}$$

__Remark:__ This is answer we are expecting, for we know $C = 2\pi r$ for $r = 3$ yields $C = 2\pi(3) = 6\pi$.

__Remark__ [added after class]: Arc length of the graph a function $y = f(x)$ can be handled as a special case of arc length of a parametric curve: take x itself to be the parameter,

$$\begin{cases} x = x \\ y = f(x) \end{cases} \quad a \le x \le b. \quad \text{Then}$$



$$\text{arc length} = \int_a^b \sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

__ex__ §8.1 #7) 7th ed. Find the length of $y = 1 + 6x^{3/2}$, $0 \le x \le 1$.

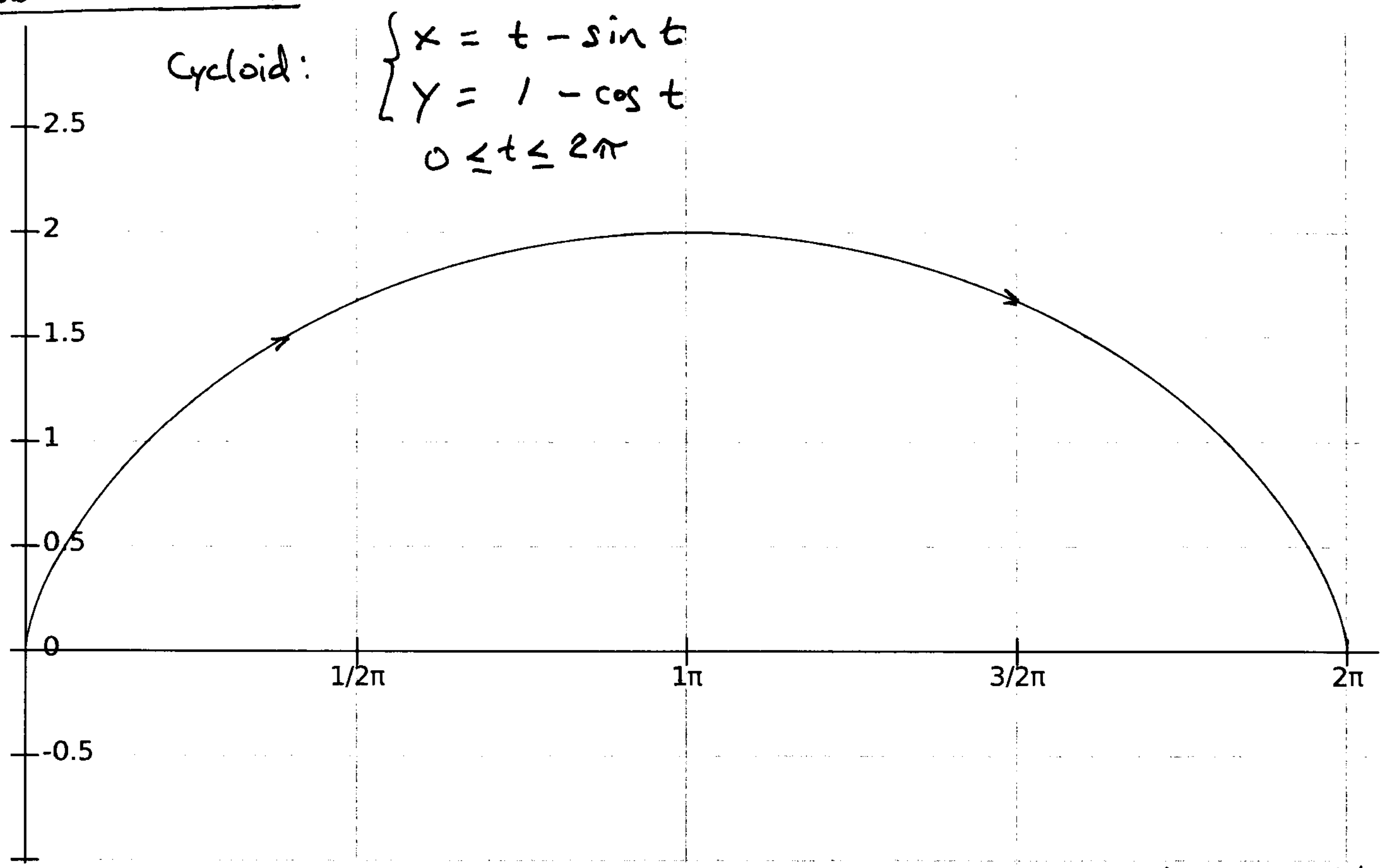
$$y' = 9x^{1/2} \Rightarrow (y')^2 = 81x \Rightarrow \sqrt{1 + (y')^2} = (1 + 81x)^{1/2}$$

$$\text{arc length} = \int_0^1 (1 + 81x)^{1/2}\, dx = \frac{2}{3}\cdot\frac{1}{81}(1 + 81x)^{3/2}\Big|_0^1 = \frac{2}{243}\left(82^{3/2} - 1\right)$$

$$= \boxed{\frac{2}{243}\left(82\sqrt{82} - 1\right)} \approx 6.103$$

Added after class

Cycloid: $\begin{cases} x = t - \sin t \\ y = 1 - \cos t \end{cases}$ $0 \le t \le 2\pi$



ex [A less trivial arclength example] Find the arclength of one arch of this cycloid.

$$\begin{cases} x = t - \sin t \\ y = 1 - \cos t \end{cases} \Rightarrow \quad \frac{dx}{dt} = 1 - \cos t \quad \Rightarrow \left(\frac{dx}{dt}\right)^2 = 1 - 2\cos t + \cos^2 t$$

$$\frac{dy}{dt} = \sin t \qquad \left(\frac{dy}{dt}\right)^2 = \sin^2 t$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 1 - 2\cos t + (\cos^2 t + \sin^2 t)$$
$$= 2 - 2\cos t = 2(1-\cos t)$$
$$= 4\left(\frac{1-\cos t}{2}\right)$$

$$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 2\sqrt{\frac{1-\cos t}{2}} = 2\sin\frac{t}{2}$$

for recall the half-angle identity $\sin\left(\frac{t}{2}\right) = \pm\sqrt{\frac{1-\cos t}{2}}$ where '+' or '-' is chosen depending on the quadrant of $t/2$. In our case we chose '+' because $\sin\frac{t}{2}$ is positive for $0 \le \frac{t}{2} \le \pi$.

$$\therefore \text{ arc length} = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} 2\sin\frac{t}{2}\, dt = -4\cos\frac{t}{2}\Big|_0^{2\pi}$$

$$= -4\cos\pi + 4\cos 0 = -4(-1) + 4(1)$$

$$= \boxed{8}$$