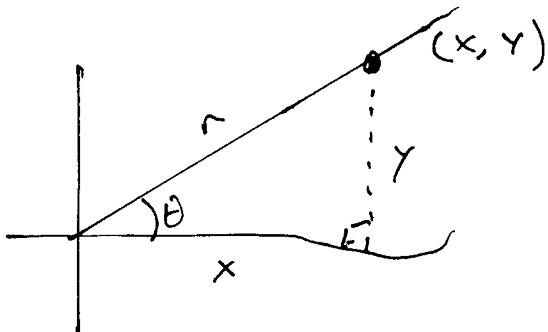
10.3 Polar coordinates



$\cos\theta = \frac{x}{r}$

$\sin\theta = \frac{y}{r}$

(In effect) Definition of polar coordinates

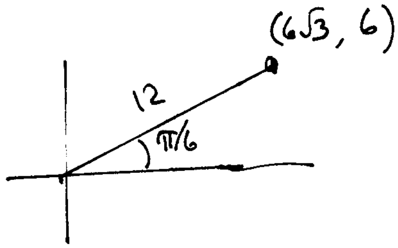
$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

$$\boxed{\begin{aligned} x^2+y^2 &= r^2 \\ \frac{y}{x} &= \tan\theta \end{aligned}}$$

$$x^2 = r^2\cos^2\theta$$
$$y^2 = r^2\sin^2\theta$$
$$x^2+y^2 = r^2\underbrace{(\cos^2\theta+\sin^2\theta)}_{1}$$

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

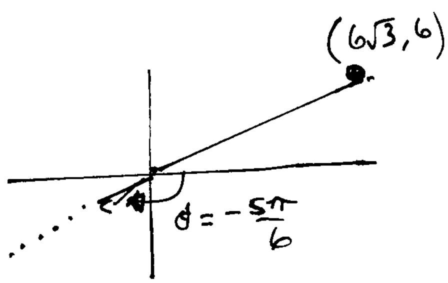
ex: For some point $r=12$, $\theta = \pi/6$. What are x and y?



$x = 12\cos\frac{\pi}{6} = 12\left(\frac{\sqrt{3}}{2}\right) = 6\sqrt{3}$

$y = 12\sin\pi/6 = 12\left(\frac{1}{2}\right) = 6$

ex: For some point $r = -12$, $\theta = \frac{-5\pi}{6}$. What are x and y?

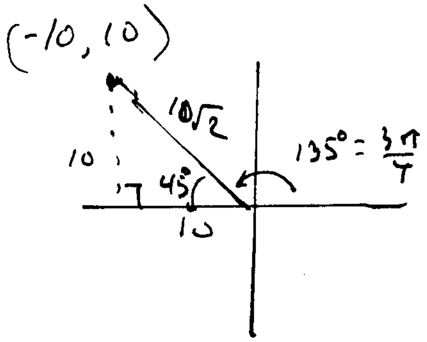


$x = -12\cos\left(\frac{-5\pi}{6}\right) = -12\left(\frac{-\sqrt{3}}{2}\right) = 6\sqrt{3}$

$y = -12\sin\left(\frac{-5\pi}{6}\right) = -12\left(-\frac{1}{2}\right) = 6$

The same point! There is a many-to-one correspondence between polar coordinates and points on the plane.

ex: The rectangular coordinates of a point are $(x,y) = (-10, 10)$. What are the polar coordinates?



$$r^2 = x^2 + y^2 = (-10)^2 + 10^2 = 200$$

So $r = \pm\sqrt{200} = \begin{cases} 10\sqrt{2} & \text{or} \\ -10\sqrt{2} \end{cases}$

Also $\dfrac{y}{x} = \dfrac{10}{-10} = -1 = \tan\theta$

$$\theta = \arctan(-1) + k\pi$$
$$= -\frac{\pi}{4} + k\pi$$

Infinitely many answers: $(r, \theta) = \left(10\sqrt{2}, \frac{3\pi}{4}\right)$ OR $\left(-10\sqrt{2}, -\frac{\pi}{4}\right)$

or $\left(10\sqrt{2}, \frac{11\pi}{4}\right)$ $\quad \left(-10\sqrt{2}, \frac{7\pi}{4}\right)$

or $\left(10\sqrt{2}, \frac{19\pi}{4}\right)$ $\quad \left(-10\sqrt{2}, \frac{15\pi}{4}\right)$

~~or~~ $\cancel{\left(10\sqrt{2}, \frac{\pi}{4}\right)}$

$\vdots$

Check: $x = r\cos\theta$, $y = r\sin\theta$

$x = 10\sqrt{2}\cos\frac{3\pi}{4} = -10$ ✓
$y = 10\sqrt{2}\sin\frac{3\pi}{4} = 10$

or $x = -10\sqrt{2}\cos\left(-\frac{\pi}{4}\right) = -10$ ✓
$y = -10\sqrt{2}\sin\left(-\frac{\pi}{4}\right) = 10$

BUT $x = +10\sqrt{2}\cos\left(-\frac{\pi}{4}\right) = 10\sqrt{2}\left(\frac{1}{\sqrt{2}}\right) = 10$ ✗
$y = 10\sqrt{2}\sin\left(-\frac{\pi}{4}\right) = 10\sqrt{2}\left(\frac{-1}{\sqrt{2}}\right) = -10$

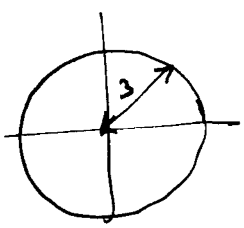
Converting equations between rectangular and polar coordinates.

ex: $x^2+y^2=9$ Convert this curve to polar coordinates.

Long way: $(r\cos\theta)^2+(r\sin\theta)^2=9$ Technically, we're done.

$$r^2\cos^2\theta+r^2\sin^2\theta=9$$
$$r^2(\cos^2\theta+\sin^2\theta)=9$$
$$r^2=9$$
$$\boxed{r=3}$$

Short way: We know: $\begin{cases} x^2+y^2=r^2 \\ \frac{y}{x}=\tan\theta \end{cases}$



$x^2+y^2=9$ Substitute r^2 for x^2+y^2:

$r^2=9 \Rightarrow r=3$

ex: $r=4\cos\theta$ Convert to rectangular coordinates.

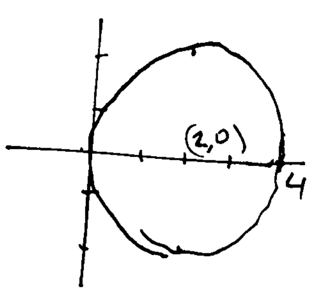
Trick: Multiply both sides by r.

$r^2=4r\cos\theta$ Use $x^2+y^2=r^2$ and $x=r\cos\theta$

$x^2+y^2=4x$ Technically we're done, but we can rewrite this to get more insight.

$$x^2-4x \quad +y^2=0$$
$$x^2-4x+4 \quad +y^2=4$$
$$(x-2)^2+y^2=4$$

circle
center $=(2,0)$
radius $=2$

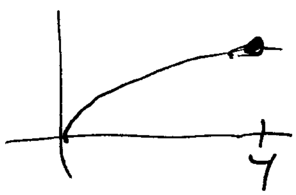


<u>Remark</u>: The following two cases can be thought of as <u>special</u> cases of parametric curves:

(1) $y = f(x)$, $a \le x \le b$ <u>Make</u> a set of parametric equations

(Let $x = t$)   by $\begin{cases} x = t \\ y = f(t) \end{cases}$ $a \le t \le b$.

<u>e.x</u>: $y = \sqrt{x}$, $0 \le x \le 4$



$\begin{cases} x = t \\ y = \sqrt{t} \end{cases}$ $0 \le t \le 4$

(2) Given $r = f(\theta)$, $\alpha \le \theta \le \beta$ namely, a <u>polar</u> curve, we can <u>also</u> make this into a parametric curve

by (let $\theta = t$) $\begin{cases} x = r\cos\theta = f(t) \cdot \cos t \\ y = r\sin\theta = f(t) \cdot \sin t \end{cases}$

<u>ex</u>: $r = 4\sin 3\theta$ ← polar curve

$\begin{cases} x = r\cos\theta = (4\sin 3t)\cos t \\ y = r\sin\theta = (4\sin 3t)\sin t \end{cases}$

<u>Remark</u>: Who cares? This means, any formula for parametric curves carries over to polar curves as a special case.

For example, we know that the slope of a tangent line to a parametric curve is $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

Hence, the slope of a tangent line to a polar curve ? Given $r = f(\theta)$

Make a parametric curve out of this.

$$\begin{cases} x = r\cos\theta = f(\theta)\cos\theta \\ y = r\sin\theta = f(\theta)\sin\theta \end{cases}$$

Then $$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

OR $$\boxed{\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}}$$

ex: Where, on the three-leaf rose $r = 4\sin 3\theta$ is the tangent horizontal? That is $\frac{dy}{dx} = 0$?

We need only show $0 = \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$

But $r = 4\sin 3\theta$

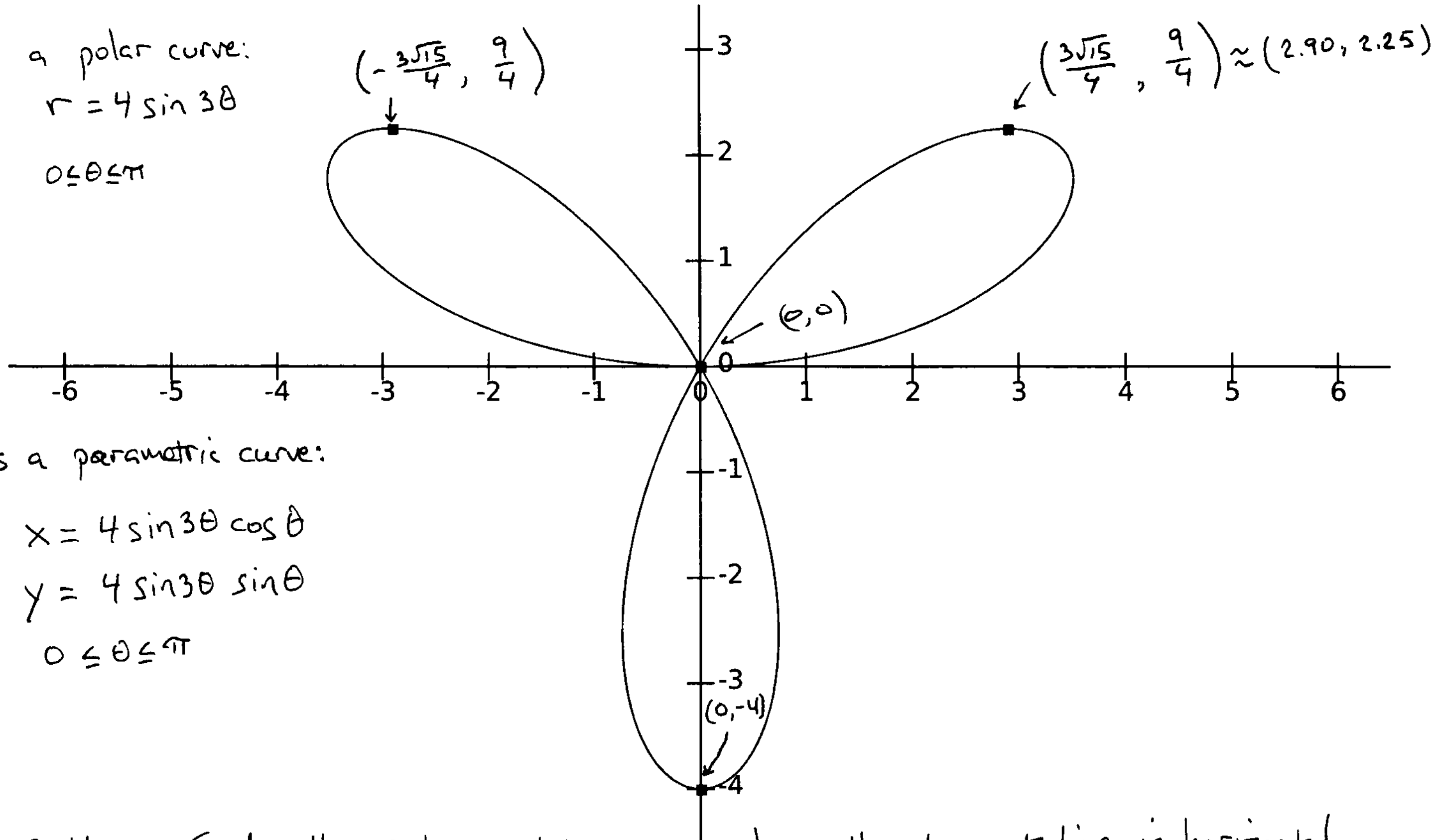
so $\frac{dr}{d\theta} = 12\cos 3\theta$

so Solve $0 = 12\cos 3\theta \sin\theta + 4\sin 3\theta\cos\theta$

[We ran out of time during class, so the following is a complete and thorough solution written after class.]

Three-leaf rose

As a polar curve:
$r = 4\sin 3\theta$
$0 \le \theta \le \pi$



As a parametric curve:

$$\begin{cases} x = 4\sin 3\theta \cos\theta \\ y = 4\sin 3\theta \sin\theta \end{cases}$$
$0 \le \theta \le \pi$

Problem: Find all points on this curve where the tangent line is horizontal.

Solution: Because $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta}$, to solve $\frac{dy}{dx} = 0$ it suffices to solve $\frac{dy}{d\theta} = 0$

It's convenient to use some trig identities to rewrite the parametric equations into an equivalent form. Using the identities ("product-to-sum")

$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$ and
$$2\sin A \sin B = \cos(A-B) - \cos(A+B)$$

with $A = 3\theta$ and $B = \theta$ so that $A+B = 4\theta$ and $A-B = 2\theta$

the parametric equations are:

$$\begin{cases} x = 4\sin 3\theta\cos\theta = 2(2\sin 3\theta\cos\theta) = 2(\sin 4\theta + \sin 2\theta) \\ y = 4\sin 3\theta \sin\theta = 2(2\sin 3\theta\sin\theta) = 2(\cos 2\theta - \cos 4\theta) \end{cases}$$

we want to solve

$$0 = \frac{dy}{d\theta} = 2\left[-2\sin 2\theta + 4\sin 4\theta\right] = 4\left[2\sin 4\theta - \sin 2\theta\right]$$
$$= 4\left[2(2\sin 2\theta\cos 2\theta) - \sin 2\theta\right] = 4\sin 2\theta\left[4\cos 2\theta - 1\right]$$

$\Rightarrow$ case 1: $\sin 2\theta = 0$

$2\theta = 0$ or π

$\theta = 0$ or $\pi/2$

$\therefore$ $(x,y) = (0,0)$ or $(0,-4)$

for $(r,\theta) = (0,0)$ or $(4, \frac{\pi}{2})$, respectively

OR case 2: $4\cos 2\theta - 1 = 0$

$\boxed{\cos 2\theta = 1/4}$ From this, we can get that $2\theta = \cos^{-1}(1/4)$ or $2\pi - \cos^{-1}(1/4)$

so $\theta = \frac{1}{2}\cos^{-1}(1/4)$ or $\pi - \frac{1}{2}\cos^{-1}(1/4)$

so $\theta \approx 37.76^\circ$ or $\theta \approx 142.24^\circ$

[continued] →

But we don't really need θ to find the (x,y)-coordinates of points where the tangent to the curve is horizontal, in this second case. Rather, it's enough to know that $\cos 2\theta = \frac{1}{4}$ at the points of tangency so that $\sin 2\theta = \pm\sqrt{1-\cos^2 2\theta} = \pm\sqrt{1-\left(\frac{1}{4}\right)^2} = \pm\frac{\sqrt{15}}{4}$.

Then, at the point of tangency, by using double-angle identities and factoring,

$$x = 2(\sin 4\theta + \sin 2\theta) = 2\{2\sin 2\theta\cos 2\theta + \sin 2\theta\}$$
$$= 2\sin 2\theta\,(2\cos 2\theta + 1) = 2\left(\pm\frac{\sqrt{15}}{4}\right)\left(2\cdot\frac{1}{4}+1\right) =$$
$$= \left(\pm\frac{\sqrt{15}}{2}\right)\left(\frac{3}{2}\right) = \pm\frac{3\sqrt{15}}{4}$$

$$y = 2(\cos 2\theta - \cos 4\theta) = 2\left[\cos 2\theta - (2\cos^2 2\theta - 1)\right]$$
$$= 2\{-2\cos^2 2\theta + \cos 2\theta + 1\} = -2\,(2\cos^2 2\theta - \cos 2\theta - 1)$$
$$= -2\,(2\cos 2\theta + 1)(\cos 2\theta - 1)$$
$$= -2\left(2\cdot\frac{1}{4}+1\right)\left(\frac{1}{4}-1\right) = -2\left(\frac{3}{2}\right)\left(-\frac{3}{4}\right) = \frac{9}{4}$$

$\therefore$ There are four points on the three-leaf rose where a tangent line to the curve will be horizontal:

$$\boxed{\begin{aligned} (x,y) &= (0,0) \\ (x,y) &= (0\ {-4}) \\ (x,y) &= \left(\frac{3\sqrt{15}}{4}, \frac{9}{4}\right) \quad \text{and} \\ (x,y) &= \left(-\frac{3\sqrt{15}}{4}, \frac{9}{4}\right). \end{aligned}}$$