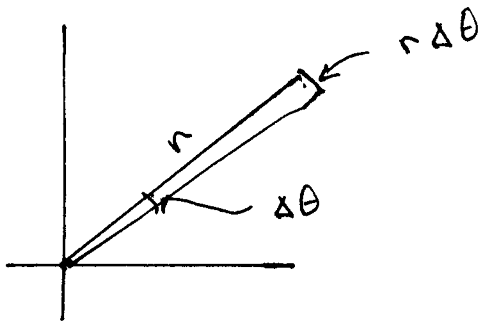
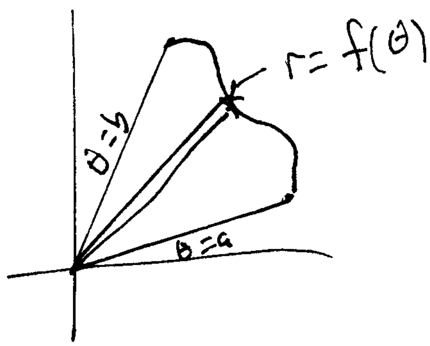
10.4 Area in polar coordinates



Area of <u>one</u> skinny triangle $\approx \frac{1}{2}(b)(h)$

$= \frac{1}{2}(r\Delta\theta)r$

$= \frac{1}{2}r^2\Delta\theta$

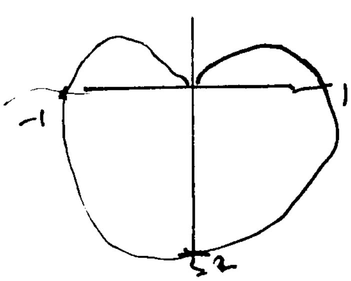


Sum of area of all triangles $= \sum_n \frac{1}{2}r^2\Delta\theta$

area $\approx \int_{\theta=a}^{\theta=b} \frac{1}{2}r^2\,d\theta = \int_a^b \frac{1}{2}[f(\theta)]^2\,d\theta$

10) $r = 1-\sin\theta$

area of cardiod $= \int_{\theta=0}^{2\pi} \frac{1}{2}r^2\,d\theta = \int_0^{2\pi} \frac{1}{2}(1-\sin\theta)^2\,d\theta$



$= \int_0^{2\pi} \frac{1}{2}\left(1-2\sin\theta+\sin^2\theta\right)d\theta$

$= \int_0^{2\pi} \frac{1}{2}\left[1-2\sin\theta+\frac{1}{2}-\frac{1}{2}\cos 2\theta\right]d\theta$

$= \int_0^{2\pi}\left(\frac{3}{4}-\sin\theta-\frac{1}{4}\cos 2\theta\right)d\theta$

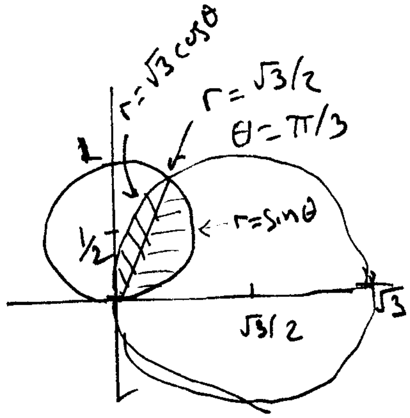
10 cont'd) $= \left[\frac{3}{4}\theta + \cos\theta - \frac{1}{8}\sin 2\theta \right]_0^{2\pi}$

$= \frac{3}{4}\{2\pi - 0\} + 0 - 0 = \boxed{\frac{3}{2}\pi}$

29) $r = \sqrt{3}\cos\theta$

$r = \sin\theta$

When graphed, these sure look like circles. Are they?



convert to rectangular:

$r = \sin\theta$	$r = \sqrt{3}\cos\theta$
$r^2 = r\sin\theta$	$r^2 = \sqrt{3}\, r\cos\theta$
$x^2+y^2 = y$	$x^2+y^2 = \sqrt{3}\,x$
$x^2+y^2-y+\frac{1}{4} = \frac{1}{4}$	$x^2-\sqrt{3}\,x+\frac{3}{4}+y^2 = \frac{3}{4}$
$x^2+\left(y-\frac{1}{2}\right)^2 = \frac{1}{4}$	$\left(x-\frac{\sqrt{3}}{2}\right)^2+y^2 = \frac{3}{4}$
center $= (0, \frac{1}{2})$	center $= (\frac{\sqrt{3}}{2}, 0)$
radius $= \frac{1}{2}$	radius $= \frac{\sqrt{3}}{2}$

where do the circles intersect?

Solve the system $\begin{cases} r = \sqrt{3}\cos\theta \\ r = \sin\theta \end{cases} \Rightarrow$

$\sqrt{3}\cos\theta = \sin\theta$

$\sqrt{3} = \frac{\sin\theta}{\cos\theta} = \tan\theta$

$\boxed{\theta = \pi/3}$ or $\frac{4\pi}{3}$ or ...

and $r = \sin\pi/3 = \boxed{\frac{\sqrt{3}}{2}} = \sqrt{3}\cos\frac{\pi}{3}$

$\left\{\begin{matrix}\text{area of} \\ \text{red region}\end{matrix}\right\} = \int_0^{\pi/3} \frac{1}{2} r^2\, d\theta = \int_0^{\pi/3} \frac{1}{2}[\sin\theta]^2 d\theta = \int_0^{\pi/3} \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right) d\theta$

$= \int_0^{\pi/3} \left(\frac{1}{4} - \frac{1}{4}\cos 2\theta\right) d\theta = \left[\frac{1}{4}\theta - \frac{1}{8}\sin 2\theta\right]_0^{\pi/3}$

$= \frac{1}{4}\left(\frac{\pi}{3}\right) - \frac{1}{8}\sin 2\cdot\frac{\pi}{3} - 0 = \frac{\pi}{12} - \frac{1}{8}\cdot\frac{\sqrt{3}}{2}$

$= \boxed{\pi/12 - \sqrt{3}/16}$

29 cont'd)

$$\left\{\begin{array}{l}\text{area of}\\ \text{blue region}\end{array}\right\} = \int_{\theta=\pi/3}^{\theta=\pi/2} \frac{1}{2} r^2 \, d\theta = \int_{\pi/3}^{\pi/2} \frac{1}{2} \left(\sqrt{3}\cos\theta\right)^2 d\theta$$

$$= \int_{\pi/3}^{\pi/2} \frac{3}{2} \cos^2\theta \, d\theta = \frac{3}{2}\int_{\pi/3}^{\pi/2} \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= \frac{3}{2}\left[\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right]_{\pi/3}^{\pi/2}$$

$$= \frac{3}{2}\left[\frac{\pi}{4} + \frac{1}{4}\cancelto{0}{\sin \pi}\right] - \frac{3}{2}\left[\frac{\pi}{6} + \frac{1}{4}\sin\frac{2\pi}{3}\right]$$

$$= \frac{3}{2}\left[\frac{\pi}{4}\right] - \frac{3}{2}\left[\frac{\pi}{6} + \frac{\sqrt{3}}{8}\right]$$

$$= \frac{3\pi}{8} - \frac{\pi}{4} - \frac{3\sqrt{3}}{16} = \frac{\pi}{8} - \frac{3\sqrt{3}}{16}$$

$$\left\{\begin{array}{l}\text{Total area}\\ \text{of both}\\ \text{regions}\end{array}\right\} = \frac{\pi}{12} - \frac{\sqrt{3}}{16} + \frac{\pi}{8} - \frac{3\sqrt{3}}{16}$$

$$= \frac{2\pi}{24} + \frac{3\pi}{24} - \frac{4\sqrt{3}}{16}$$

$$= \boxed{\frac{5\pi}{24} - \frac{\sqrt{3}}{4}}$$

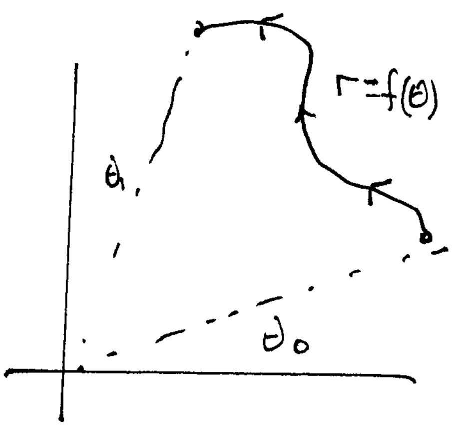
Arc length in polar coordinates

Remark Polar curves, $r=f(\theta)$, can be thought of as a special case of parametric curves: Let θ = the parameter

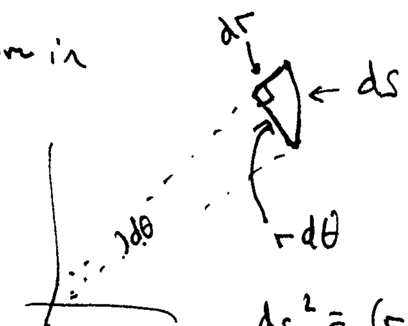
$$x(\theta) = r\cos\theta = f(\theta)\cos\theta$$
$$y(\theta) = r\sin\theta = f(\theta)\sin\theta$$

$$\text{arc length} = \int_{\theta_0}^{\theta_1} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = [\text{do algebra}]$$

$$= \boxed{\int_{\theta_0}^{\theta_1} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta}$$



Zoom in



$$ds^2 = (r\,d\theta)^2 + (dr)^2$$

element of arc length $= ds = \sqrt{r^2 d\theta^2 + dr^2}$

$$\text{Total arc length} = \int_{\theta_1}^{\theta_1} \left(\frac{ds}{d\theta}\right) d\theta = \int_{\theta_0}^{\theta_1} \sqrt{r^2\left(\frac{d\theta}{d\theta}\right)^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_{\theta_0}^{\theta_1} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

ex: Find the arc length of the cardioid

$$r = 1 - \sin\theta \qquad \frac{dr}{d\theta} = -\cos\theta$$

$$r^2 = (1-\sin\theta)^2 \qquad \left(\frac{dr}{d\theta}\right)^2 = \cos^2\theta$$

$$= 1 - 2\sin\theta + \sin^2\theta$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = 1 - 2\sin\theta + \underbrace{\sin^2\theta + \cos^2\theta}_{1}$$

$$= 2 - 2\sin\theta = 2(1-\sin\theta)$$

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{2}\sqrt{1-\sin\theta}$$

answer

$$\text{arc length} = \int_0^{2\pi} \sqrt{2}\sqrt{1-\sin\theta}\, d\theta = \boxed{8}$$

Try this: $\sin\theta = \sin 2\left(\frac{\theta}{2}\right) = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$?

$$1 = \sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}$$

$$1 - \sin\theta = \sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} - 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= \sin^2\frac{\theta}{2} - 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} + \cos^2\frac{\theta}{2}$$

$$= \left(\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\right)^2$$

$$\sqrt{1-\sin\theta} = \sin\frac{\theta}{2} - \cos\frac{\theta}{2}$$

This step is incorrect,
Correction:
$$\sqrt{1-\sin\theta} = \sqrt{\left(\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\right)^2} = \left|\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\right|$$

$$\int_0^{2\pi} \sqrt{2}\left(\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\right) d\theta = \sqrt{2}\left[-2\cos\frac{\theta}{2} - 2\sin\frac{\theta}{2}\right]_0^{2\pi}$$

$$= \sqrt{2}\left[-2\cos\pi - 2\cancelto{0}{\sin\pi}\right] - \sqrt{2}\left[-2\cos 0 - 2\cancelto{0}{\sin 0}\right]$$

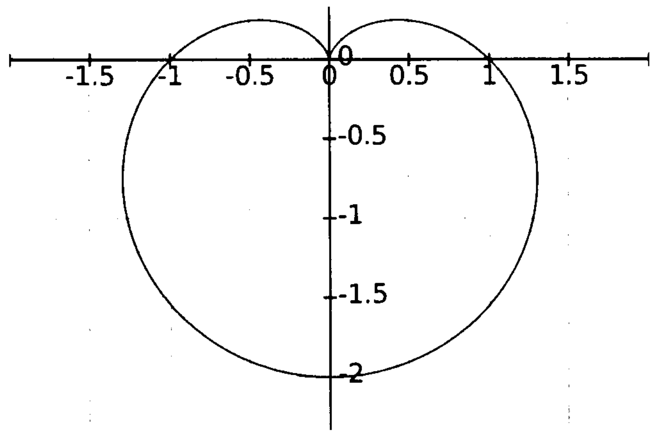
$$= \sqrt{2}\left[-2(-1)\right] - \sqrt{2}\left[-2(1)\right] = 2\sqrt{2} + 2\sqrt{2} = 4\sqrt{2}$$

Not the correct answer

example (with corrections) Find the arclength of the cardiod
[written after class]

$$r = 1 - \sin\theta \qquad \frac{dr}{d\theta} = -\cos\theta$$

$$r^2 = (1-\sin\theta)^2 = 1 - 2\sin\theta + \sin^2\theta \qquad \left(\frac{dr}{d\theta}\right)^2 = \cos^2\theta$$



$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = 1 - 2\sin\theta + \underbrace{\sin^2\theta + \cos^2\theta}_{1} = 2 - 2\sin\theta$$

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{2 - 2\sin\theta} = \sqrt{2}\sqrt{1-\sin\theta}$$

$$\boxed{\text{arclength} = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{2}\sqrt{1-\sin\theta}\, d\theta}$$

We need some trig identities: $\sqrt{2}\sqrt{1-\sin\theta} = \frac{2}{\sqrt{2}}\sqrt{1-\cos(\frac{\pi}{2}-\theta)} = 2\sqrt{\frac{1-\cos(\pi/2-\theta)}{2}}$

$$= 2\left|\sin\tfrac{1}{2}\left(\tfrac{\pi}{2}-\theta\right)\right| = 2\left|\sin\left(\tfrac{\pi}{4}-\tfrac{\theta}{2}\right)\right| = 2\left|\sin\left(\tfrac{\theta}{2}-\tfrac{\pi}{4}\right)\right|$$

[Here, we are using the Half-Angle identity $\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}}$ so that

$\left|\sin\frac{u}{2}\right| = \sqrt{\frac{1-\cos u}{2}}$. Then let $u = \frac{\pi}{2}-\theta$ so $\left|\sin\frac{1}{2}(\frac{\pi}{2}-\theta)\right| = \sqrt{\frac{1-\cos(\pi/2-\theta)}{2}}$.]

Now, $\sin\left(\frac{\theta}{2}-\frac{\pi}{4}\right)$ is positive when $0 < \frac{\theta}{2} - \frac{\pi}{4} < \pi \Rightarrow 2(0) < 2\left(\frac{\theta}{2}-\frac{\pi}{4}\right) < 2(\pi)$

$\Rightarrow 0 < \theta - \frac{\pi}{2} < 2\pi \Rightarrow \frac{\pi}{2} < \theta < 2\pi + \frac{\pi}{2} = \frac{5\pi}{2}$. The arclength integral was originally set up with limits of $\theta = 0$ and $\theta = 2\pi$, but any interval of length 2π would work as well. In particular, if we take $\frac{\pi}{2} \le \theta \le \frac{5\pi}{2}$, we can drop the absolute value symbol on $2\left|\sin\left(\frac{\theta}{2}-\frac{\pi}{4}\right)\right|$ because $\sin\left(\frac{\theta}{2}-\frac{\pi}{4}\right) \ge 0$ for those values of θ.

(cont'd)

example (cont'd) With this consideration, and using the trig identities

$$\text{arc length of the cardioid} = \int_{\pi/2}^{5\pi/2} \sqrt{2}\,\sqrt{1-\sin\theta}\, d\theta$$

$$= \int_{\pi/2}^{5\pi/2} 2\sin\left(\frac{\theta}{2}-\frac{\pi}{4}\right) d\theta$$

$$= -4\cos\left(\frac{\theta}{2}-\frac{\pi}{4}\right)\Big]_{\pi/2}^{5\pi/2}$$

$$= -4\cos\left(\frac{5\pi}{4}-\frac{\pi}{4}\right) + 4\cos\left(\frac{\pi}{4}-\frac{\pi}{4}\right)$$

$$= -4\cos(\pi) + 4\cos(0)$$

$$= -4(-1) + 4(1) = \boxed{8}$$

Remark: There are at least two or three other methods for calculating this arc length integral. None are very easy.

In particular, the following integral also produces the correct answer:

$$\text{arc length} = \int_{\pi/2}^{5\pi/2} \sqrt{2}\left(\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\right) d\theta$$

$$= \sqrt{2}\left[-2\cos\frac{\theta}{2} - 2\sin\frac{\theta}{2}\right]_{\pi/2}^{5\pi/2}$$

$$= \sqrt{2}\left[-2\cos\frac{5\pi}{4} - 2\sin\frac{5\pi}{4}\right] - \sqrt{2}\left[-2\cos\frac{\pi}{4} - 2\sin\frac{\pi}{4}\right]$$

$$= \sqrt{2}\left[-2\left(-\frac{\sqrt{2}}{2}\right) - 2\left(-\frac{\sqrt{2}}{2}\right)\right] - \sqrt{2}\left[-2\cdot\frac{\sqrt{2}}{2} - 2\frac{\sqrt{2}}{2}\right]$$

$$= \sqrt{2}\left[2\sqrt{2}\right] - \sqrt{2}\left[-2\sqrt{2}\right] = 4 + 4 = \boxed{8}$$