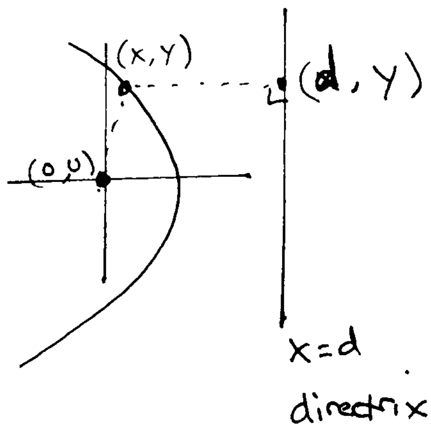
10.6 Conic Sections in polar coordinates



$$\frac{\text{origin-to-}(x,y)\text{ distance}}{\text{directrix-to-}(x,y)\text{ distance}} = e = \text{eccentricity}$$

$$\frac{\sqrt{x^2+y^2}}{d-x} = e$$

$$\sqrt{x^2+y^2} = e(d-x)$$

$$\sqrt{x^2+y^2} = ed - ex$$

Write in polar coordinates

$$r = ed - er\cos\theta$$

Solve for r:

$$r + er\cos\theta = ed$$

$$r(1+e\cos\theta) = ed$$

$$\boxed{r = \frac{ed}{1+e\cos\theta}}$$

<u>Remark</u>: If $e=0$, the conic is a <u>circle</u>.
If $e<1$, the conic is an <u>ellipse</u>.
If $e=1$, the conic is a <u>parabola</u>.
If $e>1$, the conic is a <u>hyperbola</u>.

Examples of $r = \frac{ed}{1 + e\cos\theta}$

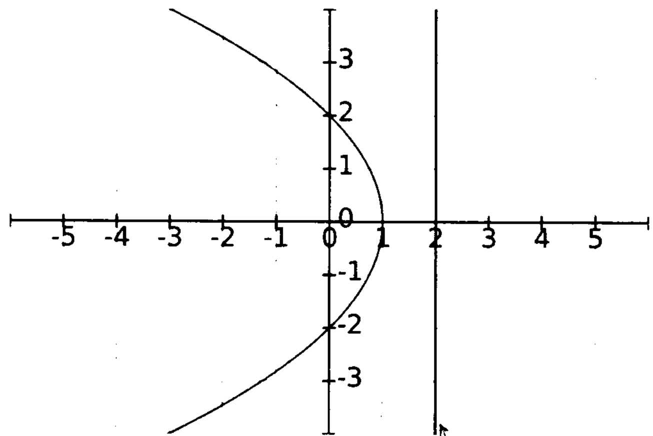
Ex 1: [Parabola] Suppose $e = 1$ and $d = 2$.

$$r = \frac{2}{1 + \cos\theta}$$

directrix (in polar form):

$$r = \frac{2}{\cos\theta}$$

Note: To get the equation of the directrix, replace the '1' with '0'.



θ	r
0	1
$\pi/2$	2
π	∞
$3\pi/2$	2

directrix: $x = 2$

focus: $(x, y) = (0, 0)$

vertex: $(x, y) = (1, 0)$

Rectangular equation: $r(1 + \cos\theta) = 2$

$$r + r\cos\theta = 2$$

$$r = -r\cos\theta + 2$$

$$r^2 = (-r\cos\theta + 2)^2 = r^2\cos^2\theta - 4r\cos\theta + 4$$

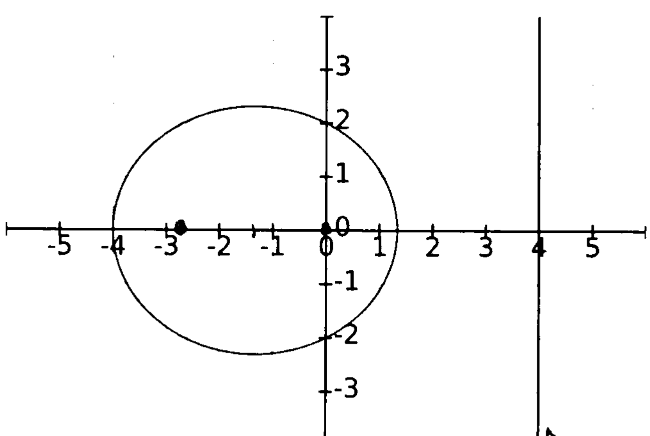
$$x^2 + y^2 = x^2 - 4x + 4$$

$$y^2 = -4(x - 1)$$

Ex 2: [ellipse] $e = \frac{1}{2}$ $d = 4$

$$r = \frac{2}{1 + \frac{1}{2}\cos\theta}$$

directrix: $r = \frac{2}{\frac{1}{2}\cos\theta} = \frac{4}{\cos\theta}$



θ	r
0	$4/3$
$\pi/2$	2
π	4
$3\pi/2$	2

$x = 4$

vertices: $(-4, 0)$, $(4/3, 0)$

foci: $(0, 0)$ $(-8/3, 0)$

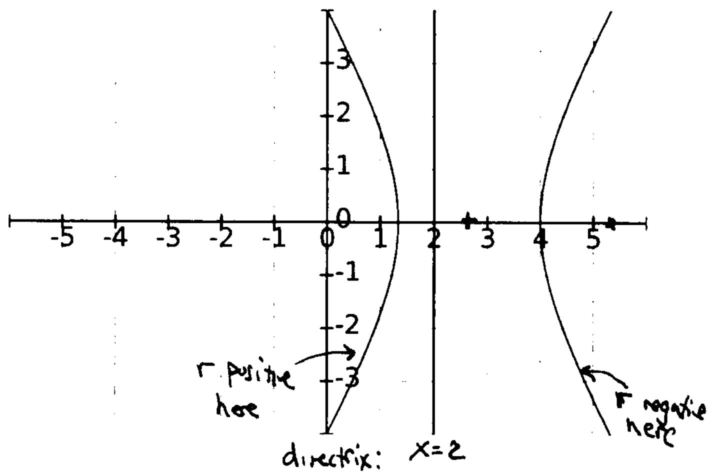
center: $(-4/3, 0)$

Ex 3: [hyperbola] Let $e = 2$ $d = 2$

$$\boxed{r = \frac{4}{1 + 2\cos\theta}}$$

directrix:

$$r = \frac{4}{2\cos\theta} = \frac{2}{\cos\theta}$$



θ	r
0	$4/3$
$\pi/2$	4
$2\pi/3$	∞
π	-4
$4\pi/3$	∞
$3\pi/2$	4

foci: $(0,0)$ $(16/3, 0)$

vertices: $(4/3, 0)$ $(4, 0)$

center: $(8/3, 0)$

slope of asymptotes: $\tan\left(\frac{2\pi}{3}\right) = \sqrt{3}$

$\tan\left(\frac{4\pi}{3}\right) = -\sqrt{3}$

equations of asymptotes: $y = \pm\sqrt{3}\left(x - \frac{8}{3}\right)$

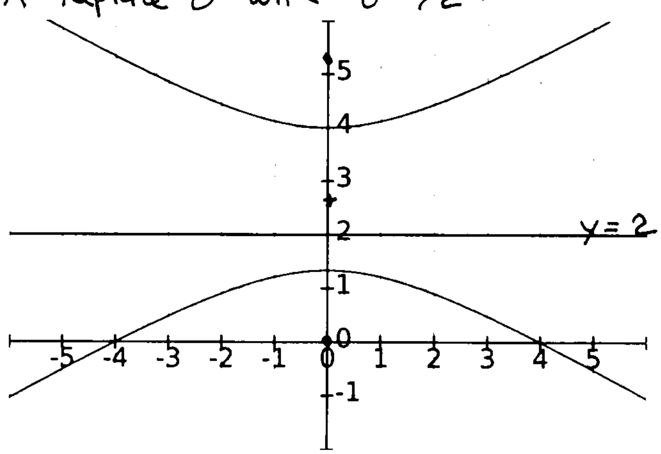
Ex 4: [Rotated hyperbola] Q: How do we rotate the polar curve by $90^\circ = \pi/2$ counter-clockwise?

A: Replace θ with $\theta - \pi/2$.

$$r = \frac{4}{1 + 2\cos(\theta - \pi/2)}$$

or $$\boxed{r = \frac{4}{1 + 2\sin\theta}}$$

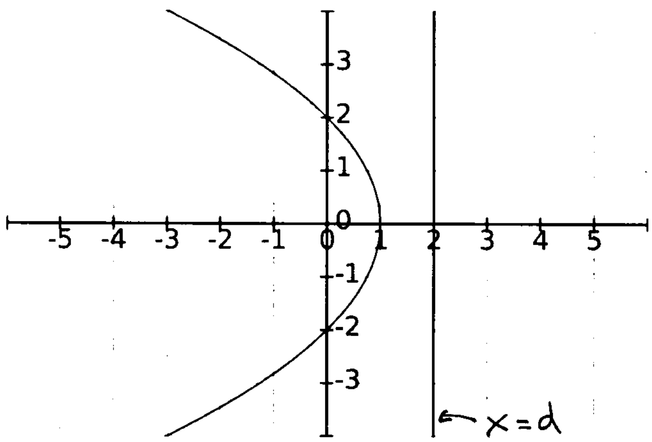
directrix: $r = \frac{2}{\sin\theta}$



θ	r
0	4
$\pi/2$	$4/3$
π	4
$7\pi/6$	∞
$3\pi/2$	-4
$11\pi/6$	∞

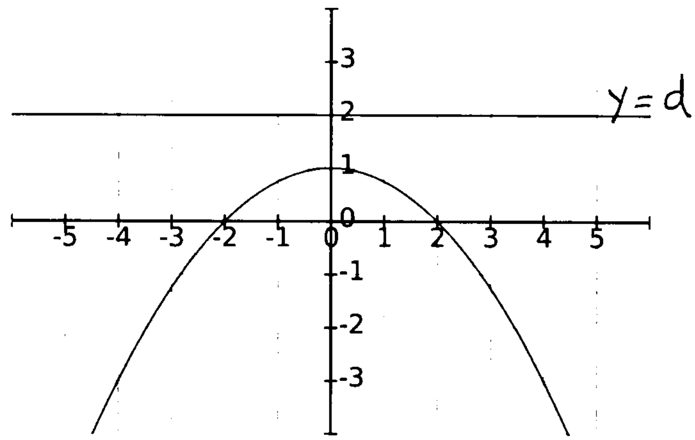
Four cases of rotated conic sections in polar form

1



$$\boxed{r = \frac{ed}{1 + e\cos\theta}}$$

2

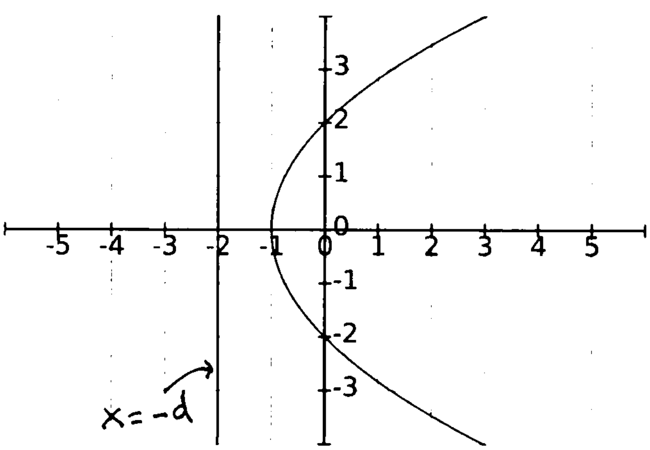


$$\boxed{r = \frac{ed}{1 + e\sin\theta}}$$

Obtained from 1 by rotating ccw by $90°$ i.e. by replacing θ with $\theta - \pi/2$ and using the identity $\cos(\theta - \frac{\pi}{2}) = \sin\theta$.

3

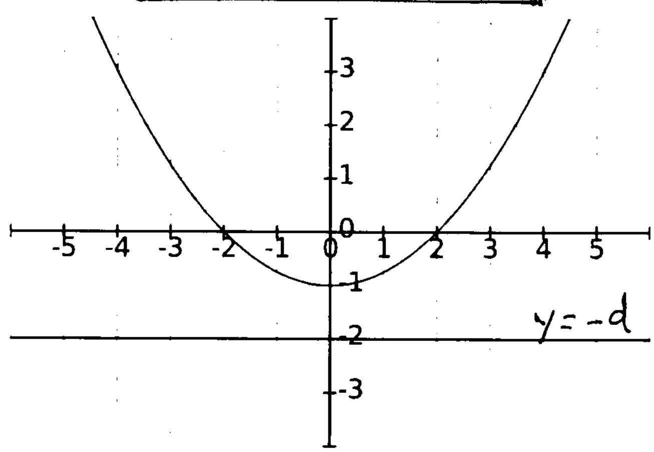
$$\boxed{r = \frac{ed}{1 - e\cos\theta}}$$



Gotten from 1 by rotating by $180°$, i.e. replace θ with $\theta - \pi$ and using $\cos(\theta - \pi) = -\cos\theta$.

4

$$\boxed{r = \frac{ed}{1 - e\sin\theta}}$$



Gotten from 2 by rotating by $180°$, i.e. replace θ with $\theta - \pi$ and using $\sin(\theta - \pi) = -\sin\theta$.

Examples added after class

[Equation → Graph]

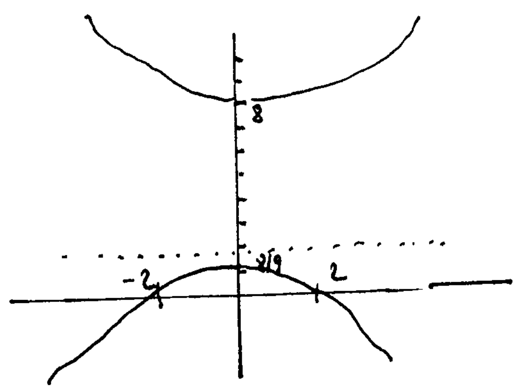
ex 10.6 # (4) For 7th ed.

$$r = \frac{8}{4+5\sin\theta} = \frac{8}{4+5\sin\theta}\cdot\frac{\frac{1}{4}}{\frac{1}{4}} = \frac{2}{1+\frac{5}{4}\sin\theta}$$

a) Find the eccentricity
b) identify the conic
c) give an eqn of the directrix
d) sketch

a) + b) By comparing with $r = \dfrac{ed}{1+e\sin\theta}$ we can see that $e = \boxed{\frac{5}{4}} > 1$ so this is a $\boxed{\text{hyperbola}}$

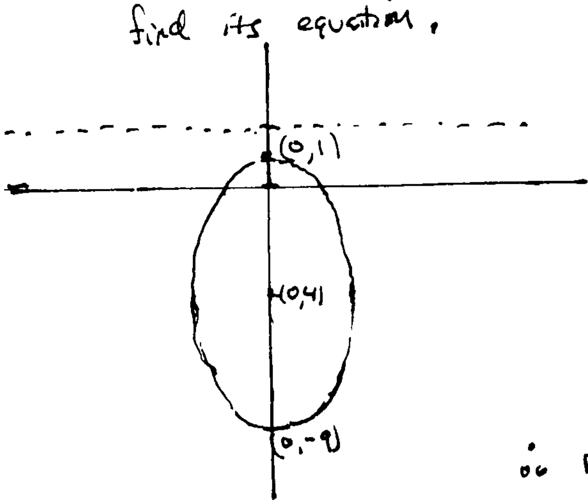
c) $d = \dfrac{ed}{e} = \dfrac{2}{5/4} = \dfrac{8}{5}$ so $r = \dfrac{8}{5\sin\theta}$ or $\boxed{y = \frac{8}{5}}$ is the directrix.



θ	r
0	2
$\pi/2$	8/9
π	2
$3\pi/2$	−8

ex [Graph → Equation]

10.6 # 6) For An ellipse with focus at origin, $e = 0.8$ and vertex closest to origin at $(r,\theta) = (1, \pi/2)$, find its equation.



Given: $e = \frac{4}{5}$ and the vertex is in the positive y-direction so the eqn. must have the form
$$r = \frac{ed}{1+e\sin\theta} \quad \text{with } e = \frac{4}{5}$$
Also when $\theta = \pi/2$, $r = 1$ so d satisfies
$$1 = \frac{\frac{4}{5}d}{1+\frac{4}{5}\sin\frac{\pi}{2}} = \frac{\frac{4}{5}d}{\frac{9}{5}} = \frac{4}{9}d \Rightarrow d = \frac{9}{4}$$
$$\therefore\ r = \frac{\left(\frac{4}{5}\right)\left(\frac{9}{4}\right)}{1+\frac{4}{5}\sin\theta} = \frac{9/5}{1+(4/5)\sin\theta} = \boxed{\frac{9}{5+4\sin\theta}}$$

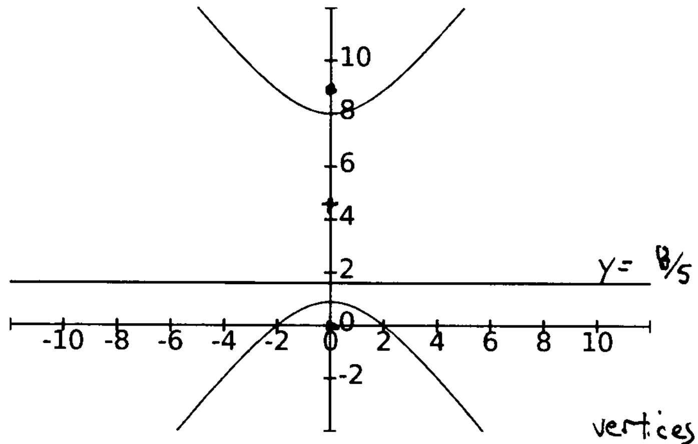
Better versions of the graphs from the previous page.

Graph for (10.6 #14)

$$r = \frac{8}{4+5\sin\theta}$$

$e = \frac{5}{4} = 1.25$

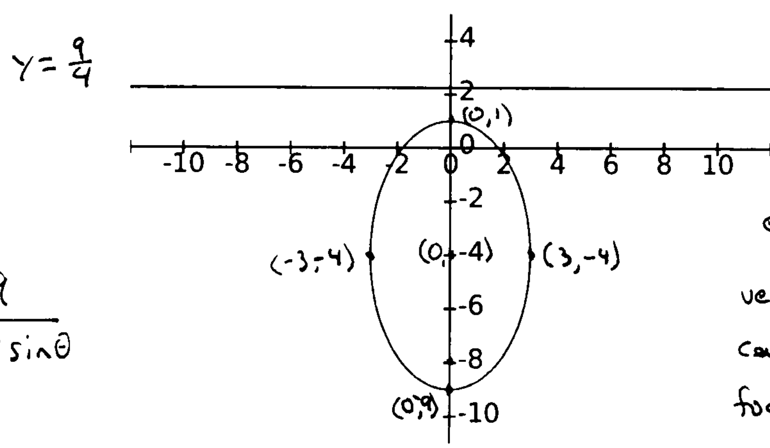
$d = \frac{8}{5}$



vertices: $(0, \frac{8}{9})$ $(0, 8)$

center: $(0, 40/9) = (0, 4\frac{4}{9})$

foci: $(0,0)$, $(0, 80/9) = (0, 8\frac{8}{9})$



$$r = \frac{9}{5+4\sin\theta}$$

$e = \frac{4}{5}$

vertices: $(0,1)$ $(0,-9)$

center: $(0,-4)$

foci: $(0,0)$ $(0,-8)$

rectangular equation: $\frac{x^2}{9} + \frac{(y+4)^2}{25} = 1$