TABULAR INTEGRATION BY PARTS

When we do integration by parts:

$$\int u\,dv = uv - \int v\,du$$

the "part" we differentiate is u, and the "part" we integrate is dv.

example: In

$$\int x \cos x \, dx$$

we take $u = x$ and $dv = \cos x\, dx$.

Then $du = dx$ and $v = \sin x$, and it follows that

$$\begin{aligned} \int x \cos x \, dx &= \int u\,dv = uv - \int v\,du \\ &= x \sin x - \int \sin x \, dx \\ &= x \sin x + \cos x + C \end{aligned}$$

The same example, in "tabular" format, proceeds as follows:

$$\begin{array}{ccccc} \underline{\text{signs}} & & \underline{\text{differentiate}} & & \underline{\text{integrate}} \\ + & \longrightarrow & x & \searrow & \cos x \\ - & \longrightarrow & 1 & \longrightarrow & \sin x \end{array}$$

It is understood that a diagonal arrow means multiply without an integral symbol, while a horizontal arrow means multiply with an integral symbol. To be precise, the integration by parts formula says that the original integral, represented as:

$$+ \longrightarrow x \longrightarrow \cos x$$

is equal to the expression represented by (note: the dx's are not written)

$$\begin{array}{ccccc} + & \longrightarrow & x & \searrow & \\ - & \longrightarrow & 1 & \longrightarrow & \sin x \end{array}$$

Tabular integration by parts is especially well-suited to cases where <u>repeated</u> integration by parts is called for.

<u>example</u>. $\int x^2 e^{5x}\,dx$ is represented by

$$+ \longrightarrow x^2 \longrightarrow e^{5x}$$

After <u>one</u> integration by parts, this is equal to

$$\begin{array}{ccccc} + & \longrightarrow & x^2 & \searrow & e^{5x} \\ - & \longrightarrow & 2x & \longrightarrow & \frac{1}{5}e^{5x} \end{array}$$

, that is,

$$(x^2)\left(\tfrac{1}{5}e^{5x}\right) - \int (2x)\left(\tfrac{1}{5}e^{5x}\right)dx$$

Applying integration by parts to the new integral

$$\begin{array}{ccccc} + & \longrightarrow & x^2 & & e^{5x} \\ - & \longrightarrow & 2x & \searrow & \frac{1}{5}e^{5x} \\ + & \longrightarrow & 2 & \longrightarrow & \frac{1}{25}e^{5x} \end{array}$$

[Because $(-1)(-1)=+1$] $\to$

says our original integral equals:

$$(x^2)\left(\tfrac{1}{5}e^{5x}\right) - (2x)\left(\tfrac{1}{25}e^{5x}\right) + \int (2)\left(\tfrac{1}{25}e^{5x}\right)dx$$

Finally, even the last integral can be done by parts <u>even</u> though it can be done without the parts formula:

$$\begin{array}{ccccc} + & \longrightarrow & x^2 & & e^{5x} \\ - & \longrightarrow & 2x & \searrow & \frac{1}{5}e^{5x} \\ + & \longrightarrow & 2 & \searrow & \frac{1}{25}e^{5x} \\ - & \longrightarrow & 0 & \longrightarrow & \frac{1}{125}e^{5x} \end{array}$$

$$(x^2)\left(\tfrac{1}{5}e^{5x}\right) - (2x)\left(\tfrac{1}{25}e^{5x}\right) + (2)\left(\tfrac{1}{125}e^{5x}\right) - \int (0)\left(\tfrac{1}{125}e^{5x}\right)dx$$

$$= (x^2)\left(\tfrac{1}{5}e^{5x}\right) - (2x)\left(\tfrac{1}{25}e^{5x}\right) + (2)\left(\tfrac{1}{125}e^{5x}\right) + C$$

Of course, there is no need to stop at every stage. The advantage of tabular integration by parts is that one can do as many integrations by parts as is needed, all at once.

<u>example</u>: $\int x^2 \sin 3x \, dx$

signs		differentiate		integrate
				$\sin 3x$
$+$	$\longrightarrow$	x^2	$\searrow$	$-\frac{1}{3}\cos 3x$
$-$	$\longrightarrow$	$2x$	$\searrow$	$-\frac{1}{9}\sin 3x$
$+$	$\longrightarrow$	2	$\searrow$	$\frac{1}{27}\cos 3x$
		0		

results in our solution

$$(x^2)\left(-\tfrac{1}{3}\cos 3x\right) - (2x)\left(-\tfrac{1}{9}\sin 3x\right) + (2)\left(\tfrac{1}{27}\cos 3x\right) + C$$

$$= -\tfrac{1}{3}x^2\cos 3x + \tfrac{2}{9}x\sin 3x + \tfrac{2}{27}\cos 3x + C .$$

Now, be warned that there are cases where repeated integration by parts is needed, but a <u>single</u> table doesn't work.
[The reason: the "parts" chosen "shift" in the second integration by parts.]

<u>example</u>: $\displaystyle\int x(\ln x)^2\,dx = (\ln x)^2\left(\frac{x^2}{2}\right) - \int\left(\frac{2\ln x}{x}\right)\left(\frac{x^2}{2}\right)dx$

$$\begin{array}{llcl} + \rightarrow & (\ln x)^2 & & x \\ - \rightarrow & \frac{2\ln x}{x} & \searrow\;\rightarrow & \frac{x^2}{2} \end{array}$$

$$= (\ln x)^2\left(\frac{x^2}{2}\right) + \int (\ln x)(-x)\,dx$$

Then $\displaystyle\int(\ln x)(-x)\,dx = (\ln x)\left(\frac{-x^2}{2}\right) - \int\left(\frac{1}{x}\right)\left(\frac{-x^2}{2}\right)dx$

$$\begin{array}{llcl} + \rightarrow & \ln x & & -x \\ - \rightarrow & \frac{1}{x} & \searrow\;\rightarrow & -\frac{x^2}{2} \end{array}$$

$$= (\ln x)\left(\frac{-x^2}{2}\right) + \int \frac{x}{2}\,dx$$

$$= (\ln x)\left(\frac{-x^2}{2}\right) + \frac{x^2}{4} + C$$

$$\therefore \int x(\ln x)^2\,dx = (\ln x)^2\left(\frac{x^2}{2}\right) + (\ln x)\left(\frac{-x^2}{2}\right) + \frac{x^2}{4} + C$$

Tabular integration by parts is even useful in cases where repeated integration by parts does not "terminate."

example: $\int e^{3x} \cos 5x \, dx$

signs		differ.		integr.
$+$	$\longrightarrow$	e^{3x}		$\cos 5x$
$-$	$\longrightarrow$	$3e^{3x}$		$\frac{1}{5}\sin 5x$
$+$	$\longrightarrow$	$9e^{3x}$	$\longrightarrow$	$-\frac{1}{25}\cos 5x$

$$\int e^{3x}\cos 5x\,dx = (e^{3x})\left(\tfrac{1}{5}\sin 5x\right) - (3e^{3x})\left(-\tfrac{1}{25}\cos 5x\right) + \int (9e^{3x})\left(-\tfrac{1}{25}\cos 5x\right)dx$$

Multiply by 25 and put the two terms containing integrals on the same side of the equal sign:

$$25\int e^{3x}\cos 5x\,dx + 9\int e^{3x}\cos 5x = 5e^{3x}\sin 5x + 3e^{3x}\cos 5x$$

or $$34\int e^{3x}\cos 5x = 5e^{3x}\sin 5x + 3e^{3x}\cos 5x$$

$$\therefore \int e^{3x}\cos 5x\,dx = \frac{5}{34}e^{3x}\sin 5x + \frac{3}{34}e^{3x}\cos 5x + C .$$

Note that this generalizes easily to allow us to state:

$$\int e^{ax}\cos bx\,dx = \frac{b}{a^2+b^2}e^{ax}\sin bx + \frac{a}{a^2+b^2}e^{ax}\cos bx + C$$

by going through the same calculation with "a" and "b" in place of 3 and 5, respectively.