Due May 10, 2011  Math 281 - Quiz 9  Name: ________________

Outside help permitted.

1. Evaluate \( \int_{C_1} \mathbf{F} \cdot d\mathbf{r} \) where \( \mathbf{F}(x, y) = (y^2 + 1, 2xy) \) and \( C_1 \) is defined by \( x = 1 - \cos t, \ y = \sin t, \) \( 0 \leq t \leq \pi, \) by three methods:
   
   a. Directly, without using that \( \mathbf{F} \) is a conservative vector field.

   \[
   \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^\pi \left( y^2 + 1 \right) dx + 2xy dy = \int_0^\pi \left[ (\sin^2 t + 1) \sin t + 2(1 - \cos t) \sin t \cos t \right] dt
   \]

   \[
   = \int_0^\pi \left[ 1 - \cos^2 t + 1 + 2 \cos t - 2 \cos^2 t \right] \sin t dt = \int_0^\pi (-\cos^2 t + 2 \cos t + 2) \sin t dt
   \]

   \[
   = \int_0^\pi (3 \cos^2 t - 2 \cos t - 2) d(\cos t) = \left[ \cos^3 t - \cos^2 t - 2 \cos t \right]_0^\pi
   \]

   \[
   = (-1 - 1 + 2) - (1 - 1 - 2) = 2
   \]

   b. Find a function \( f \) such that \( \nabla f = \mathbf{F}, \) then use the Fundamental Theorem of Line Integrals.

   \( f_x(x, y) = y^2 + 1 \) \( \Rightarrow f(x, y) = \int (y^2 + 1) dx = xy^2 + x + g(y) \) \( \Rightarrow \) we can take \( f(x, y) = xy^2 + x \)

   \( f_y(x, y) = 2xy \) \( \Rightarrow f(x, y) = \int 2xy dy = xy^2 + h(x) \)

   Then \( \nabla f = \langle f_x, f_y \rangle = \langle y^2 + 1, 2xy \rangle = \mathbf{F}. \) By the Fundamental Theorem of Line Integrals,

   \[
   \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(1)) - f(\mathbf{r}(0)) = f(2, 0) - f(0, 0)
   \]

   \[
   = (2 \cdot 0^2 + 2) - (0 \cdot 0^2 + 0) = 2
   \]

   c. Knowing that \( \mathbf{F} \) is conservative, so the integral is independent of path, evaluate \( \int_{C_2} \mathbf{F} \cdot d\mathbf{r} \)

   where \( C_2 \) is the simpler curve defined by \( x = t, \ y = 0, \ 0 \leq t \leq 2. \)

   \[
   dx = dt, \ \ dy = 0
   \]

   \[
   \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^2 (y^2 + 1) dx + 2xy dy
   \]

   \[
   = \int_0^2 (0^2 + 1) dt + 0 = t \bigg|_0^2 = 2
   \]