Determine whether San Diego lies in the **southern** half of the northern hemisphere or the **northern** half of the northern hemisphere, as follows. The latitude of San Diego is 33 degrees north of the equator. Assume the earth is a sphere of radius 1, that is, the unit of distance used is earth radii (about 6366 kilometers).

1. Let \( \mathbf{r}(u,v) = (\sqrt{1-v^2} \cos u, \sqrt{1-v^2} \sin u, v) \), \( 0 \leq u \leq 2\pi, \ 0 \leq v \leq 1 \). (This provides a parametrization for the entire northern hemisphere.)

   a. Show the rectangular equation of the surface is that of a sphere by eliminating the parameters. \( x = \sqrt{1-v^2} \cos u, \ y = \sqrt{1-v^2} \sin u, \ z = v \)

   \[ x^2 + y^2 + z^2 = (1-v^2) \cos^2 u + (1-v^2) \sin^2 u + v^2 = (1-v^2)(\cos^2 u + \sin^2 u) + v^2 = 1 - v^2 + v^2 = 1 \]

   b. Find \( |\mathbf{r}_u \times \mathbf{r}_v| \).

   \[ |\mathbf{r}_u \times \mathbf{r}_v| = \left| \begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sqrt{1-v^2} \sin u & \sqrt{1-v^2} \cos u & 0 \\ -\frac{v \cos u}{\sqrt{1-v^2}} & -\frac{v \sin u}{\sqrt{1-v^2}} & 1 \end{array} \right| = \sqrt{1-v^2} \cos u, \sqrt{1-v^2} \sin u \]

   c. Evaluate the area (in square earth radii) of the surface, \( \int \int_{D} |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv \).

   \[ \text{Area of northern hemisphere} = \int_{0}^{2\pi} \int_{0}^{1} |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv = \int_{0}^{2\pi} \int_{0}^{1} 1 \, du \, dv = \int_{0}^{2\pi} du \int_{0}^{1} dv \]

   \[ = \left[ u \right]_{0}^{2\pi} \left[ v \right]_{0}^{1} = 2\pi \text{ square earth radii} \]

2. a. Repeat exercise #1c, but with \( \mathbf{r}(u,v) = (\sqrt{1-v^2} \cos u, \sqrt{1-v^2} \sin u, v) \), \( 0 \leq u \leq 2\pi, \ 0 \leq v \leq \sin 33^\circ \). (This represents the area of the portion of the northern hemisphere which lies south of San Diego.)

   \[ \int_{0}^{\sin 33^\circ} \int_{0}^{2\pi} \, du \, dv = \left[ u \right]_{0}^{2\pi} \left[ v \right]_{0}^{\sin 33^\circ} = 2\pi \sin 33^\circ \text{ square earth radii} \]

b) What percentage of the northern hemisphere lies south of San Diego?

\[ \frac{2\pi \sin 33^\circ}{2\pi} = \sin 33^\circ \approx 0.5446 \approx 54.46\% \]

So the majority of the northern hemisphere lies south of San Diego.