16.3 (cont'd)

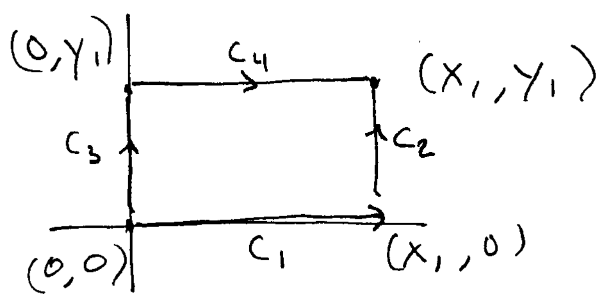
Theorem $\boxed{4}$: [...] If $\int_C \vec{F}\cdot d\vec{r}$ is independent of path, then $\vec{F}$ is conservative, i.e. $\vec{F}=\nabla f$ for some potential function $f$.

<u>example</u> to illuminate the proof.

$$\text{Let } \vec{F}(x,y) = \langle P, Q \rangle = \langle 2x\cos y, -x^2\sin y \rangle.$$

<u>Given</u>: Line integrals of $\vec{F}$ are independent of path.

<u>Task</u>: Find the potential function $f(x,y)$ so that $\nabla f = \vec{F}$ by the method of the <u>proof</u> of Theorem $\boxed{4}$.




Know $\int_{C_1} F\cdot dr + \int_{C_2} F\cdot dr$

$= \int_{C_3} F\cdot dr + \int_{C_4} F\cdot dr$

On $C_1$: $y = \text{constant} = 0$, so $dy = 0$; $0 \le x \le x_1$

$$\int_{C_1} P\,dx + Q\,\cancel{dy}^{\,0} = \int_0^{x_1} 2x\cos \cancel{y}^{\,1}\,dx = \int_0^{x_1} 2x\,dx = x^2\Big|_0^{x_1}$$

$$= x_1^2$$

On $C_2$: $x = \text{constant} = x_1$, so $dx = 0$

and $0 \le y \le y_1$

$$\int_{C_2} F \cdot dr = \int_{C_2} P\cancel{dx}^{0} + Q\,dy = \int_0^{y_1} -x_1^2 \sin y\,dy$$

$$= x_1^2 \cos y \Big|_0^{y_1} = x_1^2 \cos y_1 - x_1^2 \cancel{\cos 0}^{1}$$

$$= x_1^2 \cos y_1 - x_1^2$$

total for $C_1 + C_2$:

$\boxed{\text{version 1 of } f(x,y)}$ Take $f(x_1,y_1) = \int_{C_1} F \cdot dr + \int_{C_2} F \cdot dr = x_1^2 + (x_1^2 \cos y_1 - x_1^2)$

$$= x_1^2 \cos y_1$$

On $C_3$: $x = \text{constant} = 0$, so $dx = 0$

and $0 \le y \le y_1$.

$$\int_{C_3} \vec{F} \cdot d\vec{r} = \int_{C_3} P\cancel{dx}^{0} + Q\,dy = \int_0^{y_1} -\cancel{x^2}^{0} \sin y\,dy = 0$$

On $C_4$: $y = \text{constant} = y_1$, so $dy = 0$

and $0 \le x \le x_1$.

$$\int_{C_4} \vec{F} \cdot d\vec{r} = \int_{C_4} P\,dx + Q\cancel{dy}^{0} = \int_0^{x_1} 2x \cos y_1\,dx = x_1^2 \cos y_1$$

$\boxed{\text{version 2 of } f(x,y)}$ Take $f(x_1,y_1) = \int_{C_3} \vec{F} \cdot d\vec{r} + \int_{C_4} \vec{F} \cdot d\vec{r} = 0 + x_1^2 \cos y_1$

Remarks [See textbook proof]: By version 1 of $f(x,y)$ we get $\dfrac{\partial f(x,y)}{\partial y} = Q(x,y) = -x^2 \cos y$

By version 2 of $f(x,y)$ we get $\dfrac{\partial}{\partial y} f(x,y) = P(x,y) = 2x \cos y$