Math 252 1 Aug 2018

Remarks about Green's Theorem (not in textbook)

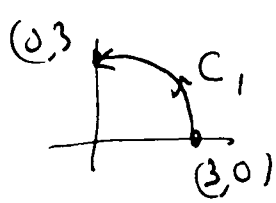
For the vector field $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ the function $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is the measure of the extent to which a line integral $\int_C P\,dx + Q\,dy$ fails to be independent of path.

We can Green's Theorem to determine how different two line integrals can be with the same start and endpoint.

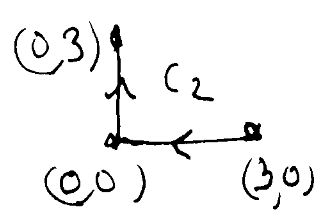
ex: let $\vec{F}(x,y) = \langle P, Q \rangle = \langle -\frac{1}{2}y, \frac{1}{2}x \rangle$

so that
$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial}{\partial x}\left(\frac{x}{2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{2}\right)$$
$$= \frac{1}{2} + \frac{1}{2} = 1 \neq 0.$$

Let C_1 be the quarter circle of radius 3:



Also, let C_2 be the piecewise linear curve:

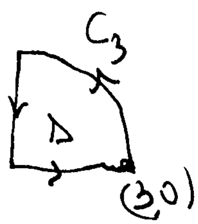


we cannot apply Green's theorem to either

$$\int_{C_1} P\,dx + Q\,dy \quad \text{or} \quad \int_{C_2} P\,dx + Q\,dy$$

because neither C_1 nor C_2 are closed curves.

TRICK: Create a new curve, C_3, by following C_1 with $-C_2$ (the same as C_2 but with opposite orientation). Then C_3 is a closed, positively oriented curve



Applying Green's Theorem we get:

$$\int_{C_3} P\,dx + Q\,dy = \int_{C_3} -\tfrac{1}{2} y\,dx + \tfrac{1}{2} x\,dy$$

$$= \iint_D 1\,dA = \text{area of } D = \frac{9\pi}{4}$$

On the other hand

$$\int_{C_3} P\,dx + Q\,dy = \int_{C_1} P\,dx + Q\,dy + \int_{-C_2} P\,dx + Q\,dy$$

$$= \int_{C_1} P\,dx + Q\,dy - \int_{C_2} P\,dx + Q\,dy$$

$$= \underbrace{\int_{C_1} -\frac{y}{2}\,dx + \frac{x}{2}\,dy}_{\frac{9\pi}{4}} - \underbrace{\int_{C_2} -\frac{y}{2}\,dx + \frac{x}{2}\,dy}_{=0}$$

In fact we could this as a techique to avoid unpleasant line integrals by replacing them with a pleasant line integral and a pleasant double integral:

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{C_1} P\,dx + Q\,dy - \int_{C_2} P\,dx + Q\,dy$$

pleasant ugly pleasant

could be rewritten as:

$$\int_{C_1} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA + \int_{C_2} P\,dx + Q\,dy$$