Math 281

a little more on 12.5 Lines and Planes

Last time: "Point - normal" form of the plane equation

$$\boxed{a(x-x_0)+b(y-y_0)+c(z-z_0)=0}$$

where $\vec{n}=\langle a, b, c\rangle$ is a normal vector to the plane, and (x_0, y_0, z_0) is a point on the plane.

"General form"

$$\boxed{ax+by+cz=d}$$

Neither form is unique for a given plane.

ex: $2x+3y+z=5$

$2(x-0)+3(y-0)+(z-5)=0$

plane through $(0,0,5)$ with normal $\vec{n}=\langle 2,3,1\rangle$.

"Intercept form"

$$\boxed{\frac{x}{x_1}+\frac{y}{y_1}+\frac{z}{z_1}=1}$$

ex: $\dfrac{x}{6}-\dfrac{y}{3}+\dfrac{z}{3}=1$ has intercepts $(6,0,0)$, $(0,-3,0)$ and $(0,0,3)$

Remark: This form fails for plane through the origin but

$\dfrac{x}{6}-\dfrac{y}{3}+\dfrac{z}{3}=0$ would be parallel to the previous plane, but would pass through the origin.

ex Take $\frac{x}{6} - \frac{y}{3} + \frac{z}{3} = 1$ and write in general form.

$$x - 2y + 2z = 6$$

which, by inspection, has a normal vector

$$\vec{n} = \langle 1, -2, 2 \rangle .$$

What if we scale the normal vector so that it has length 1?

$$|\vec{n}| = \sqrt{1^2 + (-2)^2 + 2^2} = \sqrt{9} = 3$$

So $\frac{\vec{n}}{|\vec{n}|} = \frac{1}{3}\langle 1, -2, 2 \rangle = \langle \frac{1}{3}, \frac{-2}{3}, \frac{2}{3} \rangle$ = unit vector

Equivalent plane equation:

an example of "origin-distance" form of a plane

$$\boxed{\frac{1}{3}x - \frac{2}{3}y + \frac{2}{3}z = 2}$$

Fact: If $|\vec{n}| = 1$ where $\vec{n} = \langle a, b, c \rangle$ then if $\boxed{ax + by + cz = d}$ is the equation of a plane, and $d \geq 0$, then d = distance of the plane from the origin.

Remark: The Explanation will be postponed until later in the notes.

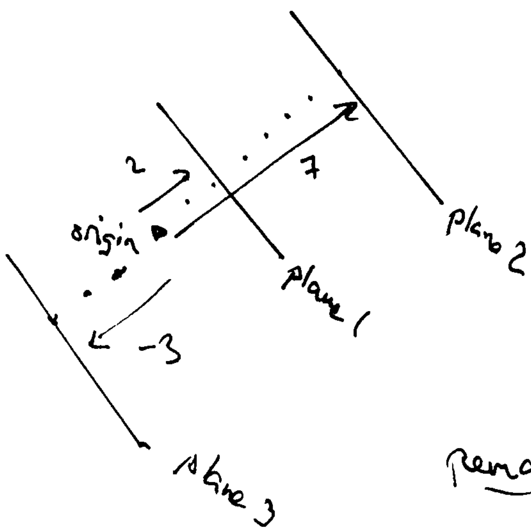
ex: Plane 1: $\frac{1}{3}x - \frac{2}{3}y + \frac{2}{3}z = 2$

plane 2: $\frac{1}{3}x - \frac{2}{3}y + \frac{2}{3}z = 7$

plane 3: $\frac{1}{3}x - \frac{2}{3}y + \frac{2}{3}z = -3$

Q1: Are the planes parallel? Yes

Q2: How far apart are planes 1 and 2? 5 units. $= 7-2$

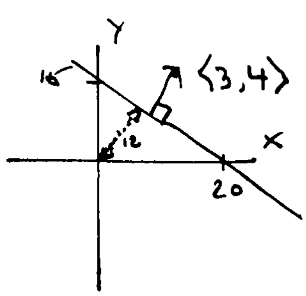


Q3: How far are planes 2 and 3?

$10 = 7-(-3)$

remarks: All these ideas work for lines in the xy-plane.

ex: $\frac{x}{20} + \frac{y}{15} = 1$ ← "Intercept form" of a line.



$3x + 4y = 60$ ← "General form"

$\vec{n} = \langle 3,4 \rangle$

Note: $|\vec{n}| = \sqrt{3^2+4^2} = 5$

$\frac{3}{5}x + \frac{4}{5}y = 12$ ← "Distance form"

The line is 12 units from the origin.

What is the closest point? $12\langle \frac{3}{5}, \frac{4}{5} \rangle = \langle \frac{36}{5}, \frac{48}{5} \rangle = \langle 7.2, 9.6 \rangle$

An alternative way to Use a TI-84 to solve the problem in Example 5, p. 845.

Ex5: Find ~~the~~ an eqn of the plane through

1st $(x, y, z) = (1, 3, 2)$

2nd $(x, y, z) = (3, -1, 6)$

3rd $(x, y, z) = (5, 2, 0)$

Idea: Expect a solution to look like

$$ax + by + cz = d$$

Choose a, b, c, and d so that the three points (ordered triples) satisfy the equation.

1st: $a + 3b + 2c = d$

2nd: $3a - b + 6c = d$

3rd: $5a + 2b = d$

Augment matrix:

$$\begin{bmatrix} 1 & 3 & 2 & 1 \\ 3 & -1 & 6 & 1 \\ 5 & 2 & 0 & 1 \end{bmatrix}$$

This matrix is row-equivalent to

$$\begin{bmatrix} 1 & 0 & 0 & 3/25 \\ 0 & 1 & 0 & 1/5 \\ 0 & 0 & 1 & 7/50 \end{bmatrix}$$

$$a = \frac{3}{25} d$$
$$b = \frac{1}{5} d$$
$$c = \frac{7}{50} d$$

we are free to choose $d = 50$.

Then $a = \frac{3}{25}(50) = 6$, $b = \frac{1}{5}(50) = 10$, $c = \frac{7}{50}(50) = 7$

Answer: $6x + 10y + 7z = 50$

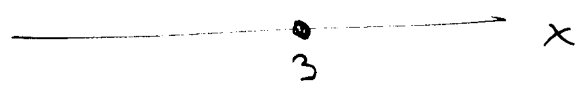
Sec. 12.6 Cylinders and Quadric surfaces

ex: What does the graph of $x = 3$ look like?

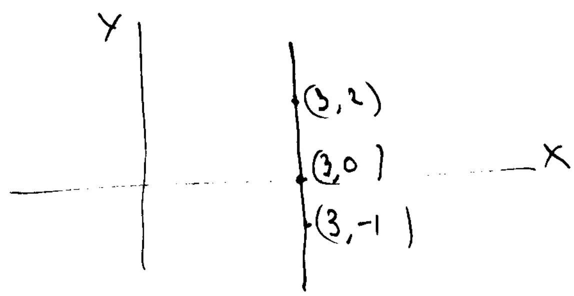
Rather: a) what does $\{x \mid x = 3\}$ look like?

b) " " $\{(x,y) \mid x = 3\}$ " " ?

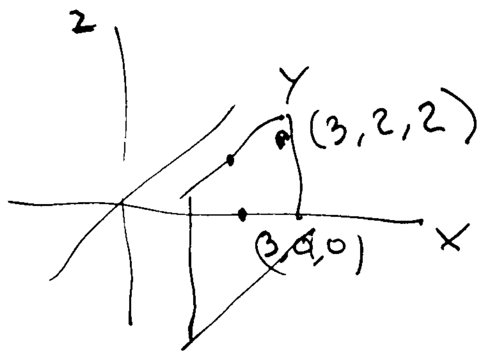
c) " " $\{(x,y,z) \mid x = 3\}$ " " ?

a) 

A point on a number line.

b) 

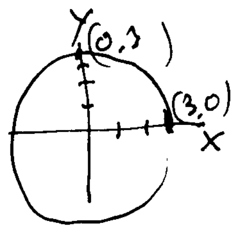
A vertical line in the xy-plane.

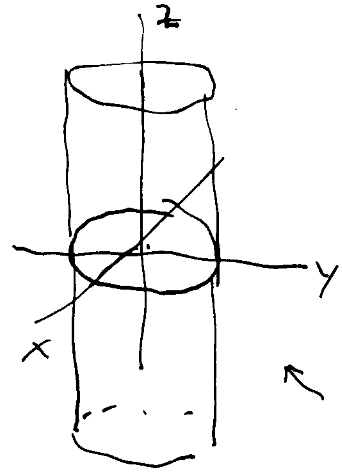
c) 

A plane in space parallel to the yz-plane.

ex: Graph $x^2+y^2=9$ in space.

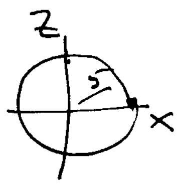
Rather $\{(x,y,z) \mid x^2+y^2=9\}$

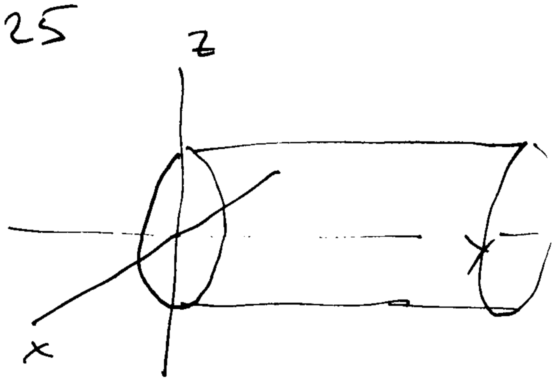




cylinder,
axis = z-axis,
radius = 3.

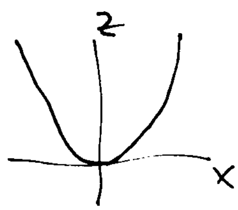
ex: $x^2+z^2=25$

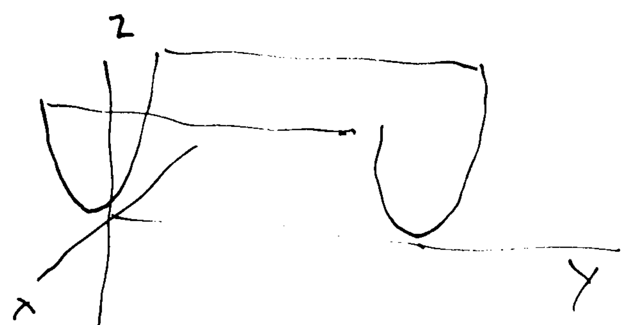




radius = 5

ex: $z=x^2$





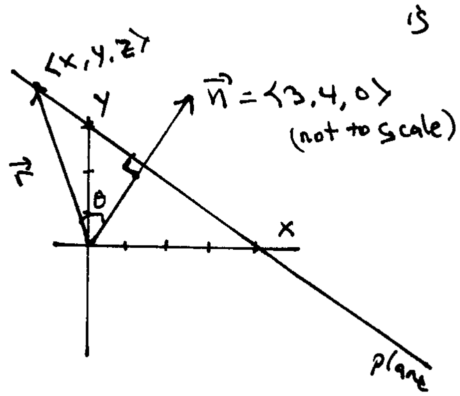
a "generalized cylinder".

Added after end of class:

An explanation for the "origin-distance" form of a plane equation:

One way to describe a plane is as the set of vectors whose scalar projection onto some fixed vector $\vec{n}$ is a (constant) number d. Note that if d is positive, this would be the plane which is d units from the origin in the direction $\vec{n}$.

For example, suppose that a plane lies 12 units from the origin in the direction $\vec{n} = \langle 3,4,0 \rangle$ and suppose (x,y,z) is a point on the plane. Then, letting



$\vec{r} = \langle x, y, z \rangle$, we have

$\text{comp}_{\vec{n}}\, \vec{r} = d = 12$. That is,

$|\vec{r}| \cos\theta = 12$ where θ = angle between $\vec{r}$ and $\vec{n}$

or $|\vec{r}| \dfrac{\vec{r}\cdot\vec{n}}{|\vec{r}||\vec{n}|} = \dfrac{\vec{r}\cdot\vec{n}}{|\vec{n}|} = \dfrac{\vec{n}}{|\vec{n}|}\cdot\vec{r} = 12$

But since $\vec{n} = \langle 3,4,0 \rangle$ we have $|\vec{n}| = \sqrt{3^2+4^2+0^2} = 5$

So that $\dfrac{\vec{n}}{|\vec{n}|} = \frac{1}{5}\langle 3,4,0\rangle = \left\langle \frac{3}{5}, \frac{4}{5}, 0 \right\rangle$.

The equation $\dfrac{\vec{n}}{|\vec{n}|}\cdot\vec{r} = 12$ becomes

$$\left\langle \frac{3}{5}, \frac{4}{5}, 0 \right\rangle \cdot \langle x, y, z \rangle = 12 \quad \text{or}$$

$$\frac{3}{5}x + \frac{4}{5}y = 12$$

More generally if $a^2+b^2+c^2 = 1$ then

"origin-distance" form of a plane

$$\boxed{ax + by + cz = d}$$

represents a plane displaced d units from the origin in the direction the plane's normal $\vec{n} = \langle a, b, c \rangle$.