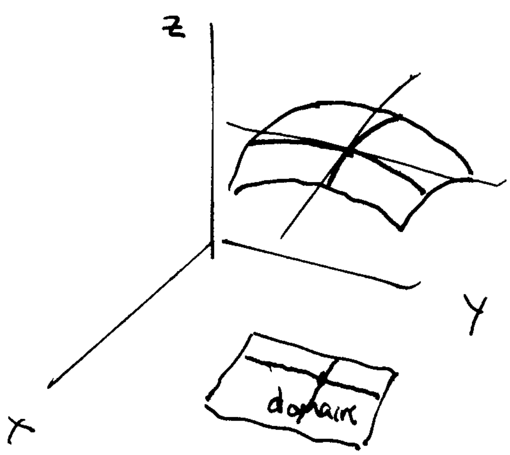
14.3 Partial derivatives (cont'd)

__Remark__: What are some interpretations for $f_x(x,y)$ and $f_y(x,y)$?

(1) $f_x(x,y)$ = instantaneous rate of change of values of f per unit of change of x __only__.

(2) $f_x(x,y)$ = slope of the line tangent to the graph of f, parallel to the xz-plane.



__Examples__ of calculating partials

24) $w = \dfrac{e^v}{u+v^2}$ Find $\dfrac{\partial w}{\partial u}$ and $\dfrac{\partial w}{\partial v}$

$$\frac{\partial w}{\partial u} = \frac{\partial}{\partial u}\left[e^v(u+v^2)^{-1}\right] = e^v \frac{\partial}{\partial u}\left[(u+v^2)^{-1}\right]$$

$$= e^v \cdot \left[-1\,(u+v^2)^{-2} \cdot \underbrace{\frac{\partial}{\partial u}[u+v^2]}_{1}\right]$$

$$= -e^v(u+v^2)^{-2}$$

$$= \frac{-e^v}{(u+v^2)^2}$$

24) (cont'd) $\frac{\partial w}{\partial v} = \frac{\partial}{\partial v}\left[\frac{e^v}{u+v^2}\right]$ Apply the quotient rule:

$$= \frac{\frac{\partial e^v}{\partial v}\cdot(u+v^2) - e^v\cdot\frac{\partial}{\partial v}[u+v^2]}{(u+v^2)^2}$$

$$= \frac{e^v(u+v^2) - e^v\cdot(2v)}{(u+v^2)^2} = \frac{e^v(u+v^2-2v)}{(u+v^2)^2}$$

32) $w = f(x,y,z) = x\sin(y-z)$

$$\frac{\partial w}{\partial x} = f_x(x,y,z) = \boxed{\sin(y-z)}$$

$$\frac{\partial w}{\partial y} = f_y(x,y,z) = \frac{\partial}{\partial y}[x\sin(y-z)] = x\frac{\partial}{\partial y}[\sin(y-z)]$$

$$= x\cos(y-z)\cdot\underbrace{\frac{\partial}{\partial y}[y-z]}_{1} = \boxed{x\cos(y-z)}$$

$$\frac{\partial w}{\partial z} = f_z(x,y,z) = \frac{\partial}{\partial z}[x\sin(y-z)] = x\frac{\partial}{\partial z}[\sin(y-z)]$$

$$= x\cos(y-z)\cdot\underbrace{\frac{\partial}{\partial z}[y-z]}_{-1} = \boxed{-x\cos(y-z)}$$

ex: For $f(x,y) = 7x^4 + 3x^2y^3 + y^5$ find $f_{xx}, f_{xy}, f_{yx}, f_{yy}$.

$f(x,y) \xrightarrow{\frac{\partial}{\partial x}} f_x(x,y) = 28x^3 + 6xy^3$

- $\xrightarrow{\frac{\partial}{\partial x}} f_{xx}(x,y) = 84x^2 + 6y^3$
- $\xrightarrow{\frac{\partial}{\partial y}} f_{xy}(x,y) = 18xy^2$

$f(x,y) \xrightarrow{\frac{\partial}{\partial y}} f_y(x,y) = 9x^2y^2 + 5y^4$

- $\xrightarrow{\frac{\partial}{\partial x}} f_{yx}(x,y) = 18xy^2$
- $\xrightarrow{\frac{\partial}{\partial y}} f_{yy}(x,y) = 18x^2y + 20y^3$

These are equal. (Not a coincidence.)

Clairaut's Theorem: Suppose that f is defined on a disk D which contains the point (a,b). If f_{xy} and f_{yx} are continuous on D, then

$$f_{xy}(a,b) = f_{yx}(a,b).$$

Remark: we have to look hard to find examples where this is NOT true. See 14.3 #101.

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

$$= \frac{(r\cos\theta)(r\sin\theta)\left[r^2\cos^2\theta - r^2\sin^2\theta\right]}{r^2}$$

$$= r^2\left[\sin\theta\cos\theta\,(\cos^2\theta - \sin^2\theta)\right]$$

$$= \tfrac{1}{2}r^2\left[\sin 2\theta \ \cos 2\theta\right] = \frac{1}{4}r^2\sin 4\theta$$

For this example $f_{xy}(0,0) \neq f_{yx}(0,0)$.

[Here we're using $x = r\cos\theta$, $y = r\sin\theta$ defines polar coordinates.]

Remark: Jumping ahead to §14.7 (p 970)

Defn: (a,b) is a critical point of f if

i) $f_x(a,b) = f_y(a,b) = 0$ or

ii) $f_x(a,b)$ doesn't exist or $f_y(a,b)$ doesn't exist

ex: $f(x,y) = (x-3)^2 + (y-4)^2$ Find critical points.

$$\begin{cases} 0 = f_x(x,y) = 2(x-3) \\ 0 = f_y(x,y) = 2(y-4) \end{cases} \Rightarrow (x,y) = (3,4).$$

14.4 Tangent Planes and Linear Approximation

Defn: If $z = f(x,y)$, then f is differentiable at (a,b)

if $z = f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$
$+ \varepsilon_1(x,y)(x-a) + \varepsilon_2(x,y)(y-b)$

where $\lim_{(x,y)\to(a,b)} \varepsilon_1(x,y) = 0 = \lim_{(x,y)\to(a,b)} \varepsilon_2(x,y)$

Remark. There is a more compact way to write this.

Let $\Delta z = f(x,y) - f(a,b)$ and $x-a = \Delta x$, $y-b = \Delta y$.

Then f is differentiable at (a,b) means

$$\Delta z = f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

Remark: Intuitively what is means for f to be differentiable is that its graph is approximated by its tangent plane.

Thm $\boxed{8}$: If f_x and f_y exist near (a,b) and are continuous at (a,b) then f is differentiable at (a,b).

Remark: This will typically be true in our examples, almost everywhere.

14.4 (cont'd)

ex: Let $f(x,y) = x^2 - y^2$ or $z = x^2 - y^2$.

Note: $z = x^2 - y^2$ is a hyperbolic paraboloid.

Show directly that $f(x,y)$ is differentiable at $(a,b) = (2,1)$.

$$z = f(x,y) = x^2 - y^2$$
$$= [(x-2)+2]^2 - [(y-1)+1]^2$$
$$= [(x-2)^2 + 4(x-2) + 2^2] - [(y-1)^2 + 2(y-1) + 1^2]$$
$$= 2^2 - 1^2 + 4(x-2) - 2(y-1) + (x-2)^2 - (y-1)^2$$

$$z = f(x,y) = 3 + 4(x-2) - 2(y-1) + (x-2)\cdot(x-2) + [-(y-1)](y-1)$$

Observe: $3 = f(2,1)$, $4 = f_x(2,1)$ [check: $f_x(x,y) = 2x$]

, $-2 = f_y(2,1)$ [check: $f_y(x,y) = -2y$]

AND if we let $\varepsilon_1 = (x-2)$ and $\varepsilon_2 = [-(y-1)]$

Then we have written

$$f(x,y) = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1) + \varepsilon_1(x-2) + \varepsilon_2(y-1)$$

so that f is differentiable at $(x,y) = (2,1)$ because $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$ as $x-2 \to 0$ and $y-1 \to 0$