

14.8 [one last topic in] Lagrange Multipliers

... maximizing $f(x, y, z)$ subject to
two constraints, $g(x, y, z) = k$ and $h(x, y, z) = c$.

Method: $\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$

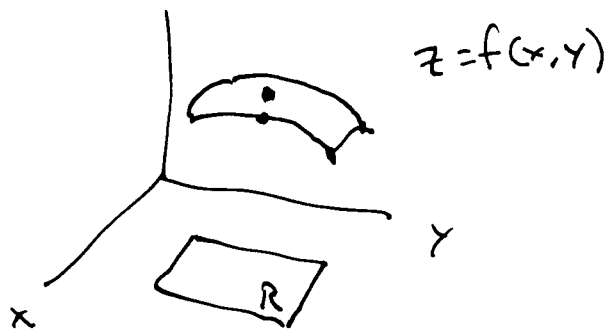
In effect,
3 eqns, 5 variables

$$\left. \begin{array}{l} g(x, y, z) = k \\ h(x, y, z) = c \end{array} \right\} \begin{array}{l} 2 \text{ more equations.} \\ \text{Total: 5 eqns} \\ \text{5 variables.} \end{array}$$

Read the book; see Example 5 p. 986. You're on your own.

Back to 14.7: Absolute max and min of a continuous $f(x, y)$
on a closed region.

Fact: If $f(x, y)$ is continuous on a closed, bounded region R
then f attains its absolute maximum and minimum.
These occur either where there is a critical point
of f or on the boundary of R .



(2)

Ex: Find the absolute maximum and minimum

$$\text{of } f(x, y) = x^2 + y^2 - 8x - 6y$$

on the closed, bounded region $R = \{(x, y) \mid x^2 + y^2 \leq 100\}$

Step 1: Unconstrained optimization

Find critical points

$$0 = f_x(x, y) = 2x - 8 = 2(x - 4) \Rightarrow (4, 3)$$

$$0 = f_y(x, y) = 2y - 6 = 2(y - 3) \quad \text{is the only C.P.}$$

Step 2: Constrained optimization

Find the max and min of $f(x, y)$ on the

circle $x^2 + y^2 = 100$, that is, the boundary of R .

Use Lagrange multipliers. Let $g(x, y) = x^2 + y^2$

$$\nabla f(x, y) = \lambda \nabla g(x, y)$$

$$\langle 2x - 8, 2y - 6 \rangle = \lambda \langle 2x, 2y \rangle$$

$$\text{so } \textcircled{1} \quad 2x - 8 = 2\lambda x$$

$$\textcircled{2} \quad 2y - 6 = 2\lambda y$$

$$\textcircled{3} \quad x^2 + y^2 = 100$$

Assume $x \neq 0$ and $y \neq 0$. Solve $\textcircled{1}$ and $\textcircled{2}$ for λ :

$$\lambda = \frac{2x - 8}{2x} = \frac{2x}{2x} - \frac{8}{2x} = 1 - \frac{4}{x} \quad \left. \vphantom{\lambda = \frac{2x - 8}{2x}} \right\} 1 - \frac{4}{x} = 1 - \frac{3}{y}$$

$$\lambda = \frac{2y - 6}{2y} = 1 - \frac{3}{y}$$

$$\Rightarrow \frac{4}{x} = \frac{3}{y} \Rightarrow 4y = 3x \Rightarrow y = \frac{3}{4}x \quad \text{Sub this into } \textcircled{3}$$

we get $x^2 + \left(\frac{3}{4}x\right)^2 = 100$

$$\frac{16x^2}{16} + \frac{9x^2}{16} = 100$$

$$\frac{25x^2}{16} = 100 \Rightarrow x^2 = \frac{16 \cdot 100}{25} = 64$$

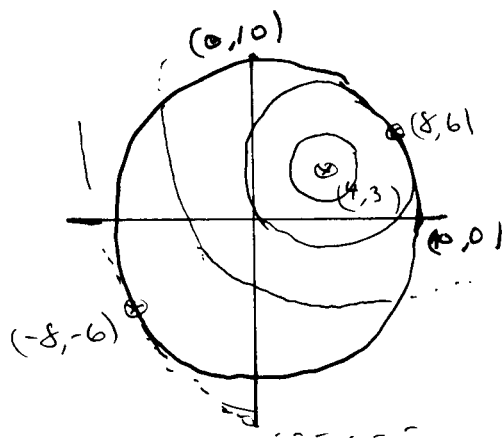
$$x = \pm 8$$

$$y = \frac{3}{4}(\pm 8) = \pm 6$$

So any (constrained) max or min happens at

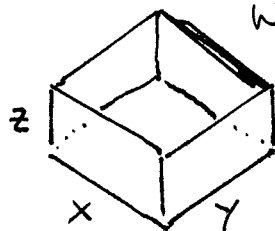
$$(x, y) = (8, 6) \text{ or } (x, y) = (-8, -6)$$

(x, y)	$f(x, y) = x^2 + y^2 - 8x - 6y$
$(4, 3)$	$f(4, 3) = 16 + 9 - 32 - 18 = -25 = \text{abs. min.}$
$(8, 6)$	$f(8, 6) = 64 + 36 - 64 - 36 = 0$
$(-8, -6)$	$f(-8, -6) = 64 + 36 + 64 + 36 = 200 = \text{abs. max.}$



ex: [A word problem involving max/min of a two variable function]

§14.7 51) A cardboard box without a lid is to have a volume of $32,000 \text{ cm}^3$. Find the dimensions which minimize the amount of cardboard used.



Method 1: Not using Lagrange multipliers

Let $f(x, y, z) = \text{area of cardboard used in the box}$
 $= xy + 2xz + 2yz$

But, since $\text{Volume} = xyz = 32,000$, we can say that $z = \frac{32,000}{xy}$. So, we instead

$$\text{let } f(x, y) = xy + 2x\left(\frac{32,000}{xy}\right) + 2y\left(\frac{32,000}{xy}\right)$$

$$= xy + 64,000 y^{-1} + 64,000 x^{-1}$$

[Note that the domain of f is $\{(x, y) : x > 0 \text{ and } y > 0\}$.]

Find critical points of f :

$$(1) \quad 0 = f_x(x, y) = y - 64,000 x^{-2} \quad \Rightarrow \quad y = \frac{64,000}{x^2}$$

$$(2) \quad 0 = f_y(x, y) = x - 64,000 y^{-2} \quad \Rightarrow \quad \text{So } y^{-2} = \frac{x^4}{64,000^2}$$

Sub into (2):

$$0 = x - 64,000 \left(\frac{x^4}{64,000^2} \right) \quad \text{Divide by } x \text{ and simplify.}$$

$$0 = 1 - \frac{x^3}{64,000} \quad \Rightarrow \quad x^3 = 64,000$$

$$x = \sqrt[3]{64,000} = 40 \text{ cm} \quad \Rightarrow \quad z = \frac{32,000}{(40)(40)}$$

$$\text{So } y = \frac{64,000}{40^2} = 40 \text{ cm} \quad = 20 \text{ cm}$$

Remark: One can apply the 2nd derivative test to confirm this is a minimum.

The same problem by method 2: Lagrange multipliers.

$$\text{Let } f(x, y, z) = xy + 2xz + 2yz$$

$$\text{Constant: volume} = xyz = 32,000 \text{ cm}^3$$

$$\text{Let } g(x, y, z) = xyz$$

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

$$\langle y+2z, x+2z, 2x+2y \rangle = \lambda \langle yz, xz, xy \rangle$$

①

$$y+2z = \lambda yz$$

②

$$x+2z = \lambda xz$$

③

$$2x+2y = \lambda xy$$

④

$$xyz = 32,000$$

Subtract ① - ②:

$$\begin{aligned} y-x &= \lambda z(y-x) \\ 0 &= \lambda z(y-x) - 1(y-x) \\ 0 &= (\lambda z - 1)(y-x) \quad \text{so} \end{aligned}$$

$$y-x=0 \quad \text{or} \quad \lambda z - 1 = 0$$

$$\text{Case 1: } \boxed{y=x}$$

$$\text{or Case 2: } \boxed{\lambda z = 1}$$

Sub into ③:

$$2x+2x = \lambda x \cdot x$$

$$4x = \lambda x^2$$

$$4 = \lambda x$$

Sub into ②:

$$x+2z = 4z$$

$$\boxed{x = 2z}$$

and since $x=y$,

$$\boxed{y = 2z}$$

Sub into ④:

$$(2z)(2z)z = 32,000$$

$$4z^3 = 32,000$$

$$z^3 = 8,000$$

$$\boxed{z = 20 \text{ cm}}$$

so

$$\boxed{x=y=2z=40 \text{ cm}}$$

Sub into ①:

$$y+2z = 1y$$

$$2z = 0$$

$$z = 0$$

Impossible, by ④,

for if $z=0$ then

$$xyz = 0 \neq 32,000$$

15.1 Double Integrals over Rectangles

Idea of Defn:

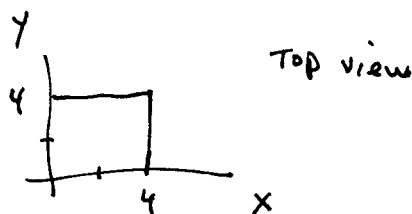
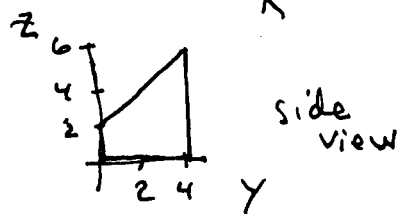
$$\iint_R f(x,y) dA \approx \sum_i \sum_j f(x_{ij}, y_{ij}) \Delta A$$

= sum of the volume of many skinny boxes

where ΔA = area of the base of the skinny box

ex: Evaluate the double integral by first identifying it as the volume of a solid

$$\iint_R (2+y) dA, \quad R = \{(x,y) \mid 0 \leq x \leq 4, 0 \leq y \leq 4\}$$



Answer: $\iint_R (2+y) dA = 64$

Defn: The average value of $f(x,y)$ on a region R is

$$f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x,y) dA$$

so that $A(R) \cdot f_{\text{ave}} = \iint_R f(x,y) dA$ where $A(R)$ = area of R

in previous
ex

$$f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x,y) dA = \frac{1}{4 \cdot 4} \cdot 64 = \frac{1}{16} (64) = 4$$

15.2 Iterated integrals

ex: Add these numbers:

1	2	4	8	15
2	4	8	16	30
3	6	12	24	45
4	8	16	32	60
10	20	40	80	150

$$\text{ex: a) } \int_0^2 x y^2 dx$$

$$= y^2 \int_{x=0}^{x=2} x dx$$

$$= y^2 \left[\frac{x^2}{2} \right]_{x=0}^2$$

$$= y^2 \left[\frac{2^2}{2} - \frac{0^2}{2} \right]$$

$$= 2y^2$$

$$\text{b) } \int_0^3 x y^2 dy$$

$$= x \int_{y=0}^{y=3} y^2 dy$$

$$= x \left[\frac{y^3}{3} \right]_0^3$$

$$= x \left[\frac{3^3}{3} - \frac{0^3}{3} \right]$$

$$= 9x$$

[Notes completed after class ended.]

(6) of 6

examples of an
"iterated integral":

$$c) \int_0^3 \int_0^2 xy^2 dx dy = \int_0^3 2y^2 dy \quad \text{by part a)}$$

$$= \frac{2}{3} y^3 \Big|_0^3 = \frac{2}{3} \cdot 3^3 = 18$$

$$d) \int_0^2 \int_0^3 xy^2 dy dx = \int_0^2 9x dx \quad \text{by part b) above}$$

$$= \frac{9}{2} x^2 \Big|_0^2 = \frac{9}{2} \cdot 2^2 = 18$$

Remark: To the extent that integrals are like sums,

$\int_0^3 xy^2 dy$ from part b) is like calculating
a subtotal, one for each y (column),
by calculating a family of integrals (sums).

Then in part d) $\int_0^2 \int_0^3 xy^2 dy dx = \int_0^2 9x dx$ is like adding all
the subtotals to get a grand total.