December 7, 1998 Math 252 Russell La Puma

Some Tips on Parametrizing Surfaces

1. For surfaces of the form $z = g(x,y)$ [or $x = g(y,z)$ or $y = g(x,z)$], take the parameters to be equal to the *independent* variables, so $\mathbf{r}(x,y) = \langle x,y,g(x,y)\rangle$,
$\mathbf{N}\,dS = (\mathbf{r}_x \times \mathbf{r}_y)\,dx\,dy = \langle -g_x, -g_y, 1\rangle\,dx\,dy$, and $dS = \|\mathbf{r}_x \times \mathbf{r}_y\|\,dx\,dy = \sqrt{g_x^{\,2} + g_y^{\,2} + 1}\,dx\,dy$.

Example 1: Parametrize the graph of $z = x^2 - y^2$, $0 \le x \le 2$, $0 \le y \le 3$.

Take $\mathbf{r}(x,y) = \langle x,y,x^2 - y^2\rangle$. Then $\mathbf{r}_x \times \mathbf{r}_y = \langle -2x, 2y, 1\rangle$ and $dS = \sqrt{4x^2 + 4y^2 + 1}\,dx\,dy$.

2. For surfaces of revolution $x^2 + y^2 = g(z)^2$ let $u = \theta$, and $v = z$ be the parameters, so that $\mathbf{r}(u,v) = \langle g(v)\cos u, g(v)\sin u, v\rangle$, $\mathbf{r}_u \times \mathbf{r}_v = g(v)\langle \cos u, \sin u, -g'(v)\rangle$, $dS = g(v)\sqrt{1 + g'(v)^2}\,du\,dv$.

Example 2: Parametrize the cone $x^2 + y^2 = 4z^2$, $0 \le z \le 3$.

Here $g(z) = 2z$. Take $\mathbf{r}(u,v) = \langle 2v\cos u, 2v\sin u, v\rangle$, $0 \le u \le 2\pi$, $0 \le v \le 3$. Then $\mathbf{r}_u \times \mathbf{r}_v = 2v\langle \cos u, \sin u, -2\rangle$ and $dS = 2\sqrt{5}\,v\,du\,dv$.

3. For surfaces of revolution written in cylindrical coordinates as $z = g(r)$, let $u = r$ and $v = \theta$ be the parameters, so $\mathbf{r}(u,v) = \langle u\cos v, u\sin v, g(u)\rangle$, $\mathbf{r}_u \times \mathbf{r}_v = u\langle -g'(u)\cos v, -g'(u)\sin v, 1\rangle$, $dS = u\sqrt{g'(u)^2 + 1}\,du\,dv$.

Example 3: Parametrize the paraboloid $z = x^2 + y^2$, $0 \le z \le 4$.

The paraboloid has the form $z = r^2$, so $g(r) = r^2$. Take $\mathbf{r}(u,v) = \langle u\cos v, u\sin v, u^2\rangle$, $0 \le u \le 2$, $0 \le v \le 2\pi$. Then $\mathbf{r}_u \times \mathbf{r}_v = u\langle -2u\cos v, -2u\sin v, 1\rangle$, $dS = u\sqrt{4u^2 + 1}\,du\,dv$.

4. To parametrize a sphere $x^2 + y^2 + z^2 = a^2$, borrow from spherical coordinates and let $u = \phi$, $v = \theta$. Let $\mathbf{r}(u,v) = \langle a\sin u\cos v, a\sin u\sin v, a\cos u\rangle$. Then $\mathbf{r}_u \times \mathbf{r}_v = a^2\sin u\langle \sin u\cos v, \sin u\sin v, \cos u\rangle$, and $dS = a^2\sin u\,du\,dv$.

Example 4: Parametrize the sphere $x^2 + y^2 + z^2 = 16$.

Take $\mathbf{r}(u,v) = \langle 4\sin u\cos v, 4\sin u\sin v, 4\cos u\rangle$, $0 \le u \le \pi$, $0 \le v \le 2\pi$. Then $\mathbf{r}_u \times \mathbf{r}_v = 16\sin u\langle \sin u\cos v, \sin u\sin v, \cos u\rangle$, $dS = 16\sin u\,du\,dv$.